

I. Consider the following three functions

$$a) r = \frac{4}{1 - \sin \theta}, \quad b) r = \frac{15}{3 - 2\cos \theta}, \quad c) r = \frac{32}{3 + 5 \sin \theta}$$

- 1) (30 points) For each of the above three equations, you will need a copy of the final page of this lab. At the top of the page write down one of the polar curves. At the bottom of the page, fill out the given chart to one decimal place. Then you will need to carefully plot the points and sketch out the curve you obtain.
- 2) (15 points) Each of the graphs you obtained in (1) should look familiar. Use the identities

$$r^2 = x^2 + y^2, \quad x = r \cos \theta, \quad y = r \sin \theta$$

to find a Cartesian equation for each of the polar equations you graphed in (1).

II. We will define a new curve C geometrically. Let F be a fixed point and l a fixed line in the plane. We will call F the focus and l the directrix. Let e be a fixed positive number that we will call the eccentricity. Define the curve C to be the set of all points P such that the ratio of the distance from P to F and the distance from P to l is the constant e . In other words, C is the set of all points P that satisfy:

$$\frac{|PF|}{|Pl|} = e$$

- 3) (10 points) If we let the focus F be the origin and the directrix l be the vertical line $x = d$, show that the curve C is represented by

$$r = \frac{ed}{1 + e \cos \theta}$$

- 4) (18 points) On a single sheet of graph paper, sketch the following 6 polar equations. Be sure to graph each equation on its own separate polar axis. Each polar axis should use the same scale. You do NOT have to plot all of the points like you did in part (I). You MAY use desmos.com or a graphing calculator to help you draw the correct picture. For each graph indicate the eccentricity and the directrix.

$$a) r = \frac{3}{1 + \frac{3}{2} \cos \theta}$$

$$b) r = \frac{\frac{5}{2}}{1 + \frac{5}{4} \cos \theta}$$

$$c) r = \frac{2}{1 + \cos \theta}$$

$$d) r = \frac{\frac{3}{2}}{1 + \frac{3}{4} \cos \theta} \quad e) r = \frac{1}{1 + \frac{1}{2} \cos \theta} \quad f) r = \frac{\frac{1}{2}}{1 + \frac{1}{4} \cos \theta}$$

- 5) (5 points) Observe that all of the functions you graphed in (4) are of the form described in (3). In fact, they all have the same directrix. The only way in which they differ from each other is by their eccentricity. Make a conjecture about how the eccentricity affects the shape of the curve C. Use a graphing device to test your conjecture and convince yourself that it is accurate.

III. Return to the three polar equations you graphed in section (I). Notice that none of these have the same form as the functions described in section (II). However they can also be shown to match the description of the curve C with the focus at the origin and an appropriate choice of a directrix.

- 6) (6 points) Consider the following three polar equations

$$a) r = \frac{ed}{1 - e \cos \theta} \quad b) r = \frac{ed}{1 - e \sin \theta} \quad c) r = \frac{ed}{1 + e \sin \theta}$$

For what directrix do each of these represent the curve C?

- 7) (6 points) Based on the work you did in (6) determine the eccentricity and directrix for the curves you graphed in (1)

IV. The constants in the Cartesian equation of a conic section have particular meanings. For instance for an ellipse, in $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, a and b represent half the length of the major and minor axes. It is important to be able to represent these quantities in terms of the eccentricity and the directrix.

- 8) (10 points) Using the equation in (3) and assuming the equation represents an ellipse, prove that

$$a = \frac{ed}{1 - e^2} \quad \text{and} \quad b = \frac{ed}{\sqrt{1 - e^2}}$$

Hint: Manipulate the equation in (3) and use the identities in (2) to obtain an equation of the form:

$$\frac{(x - h)^2}{a^2} + \frac{y^2}{b^2} = 1$$

