Ch 3: Probability

A probability is the long range relative frequency of an outcome.

Example: Suppose we flip a coin several times. We are interested in how many times we get heads.

As we flip more & more times, the rel. freq. "stabilizes" at about $\frac{1}{2}$ or 0.5 or 50%

So the probability of getting a heads is $\frac{1}{2}$

Probability Terms

- **An experiment is** an activity conducted in controlled circumstances

ex) coin flip, measure a person's height

- **An outcome is** a result of an experiment

ex) heads, tails, 5'9"

- **An event is** a set of outcomes

ex) 53 heads and 43 tails

- **The sample space is** all possible outcomes for an experiment

ex) {heads, tails}

~~S = students in this class~~
~~C = students at DA with cats~~
~~D = students at DA with dogs~~

Combining Events

- **A and B**
  - S and C = students in this class who own a cat
  - C and D = students who own a cat and a dog.

* must meet both cond.

- **A or B**
  - S or C = students who are in this class or who own a cat (1 or the other or both)

* must meet 1 condition

- **A' = Complement of A**
  - S' = students not in this class
  - C' = students who don't have a cat

"not A"

Example: S = students in this class
C = students who own a cat at DA
D = students who own a dog at DA

Calculating Probabilities

Example: Consider the experiment of rolling a yellow and an orange die and recording the faces showing. Consider the following events:

7 = rolling a sum of 7

6 = rolling a sum of 6 → {(1,5), (2,4), (3,3), (4,2), (5,1)}

D = rolling doubles

P(7) = $\frac{6}{36}$

P(6) = $\frac{5}{36}$

P(D) = $\frac{6}{36}$

P(7') = $\frac{30}{36} = \frac{36}{36} - \frac{6}{36}$

prob. of not summing to 7

P(2) = $\frac{1}{36}$

***Complement Rule:* P(A') =** $1 - P(A)$

ex) $P(6') = \frac{36}{36} - \frac{5}{36} = \frac{31}{36}$

P(6 and D) = $\frac{1}{36}$

P(7 and D) = $\frac{0}{36} = 0$

***Mutually Exclusive:* Two events A and B are mutually exclusive if** $P(A \text{ and } B) = 0$

i.e. they cannot happen at the same time

P(6 or D) = $\frac{10}{36}$

ck: $\frac{5}{36} + \frac{6}{36} - \frac{1}{36} = \frac{10}{36}$ ✓

***Addition Rule:* P(A or B) =** $P(A) + P(B) - P(A \text{ and } B)$

Note: Probabilities are always between 0 & 1 i.e. $0 \leq P(\ldots) \leq 1$

Conditional Probability $P(A|B)$

"probability of A given B"

= prob. that A will happen if you already know that B will definitely happen

Using yesterdays example...

Conditional Probability Rule

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Contingency Table

Example (HW #17 in textbook):

	Brown	Blond	Black	Red	Total
Wavy	20	5	15	3	43
Straight	80	15	65	12	172
Total	100	20	80	15	215

P(Wavy) $= \frac{43}{215}$

P(Brown or Blond) $= \frac{120}{215}$

P(Wavy and Brown) $= \frac{20}{215}$

P(Red | Straight) $= \frac{12}{172}$ or $\frac{P(\text{red and str.})}{P(\text{str.})} = \frac{12/215}{172/215}$

$= \frac{12}{172}$

P(not Brown) $= \frac{115}{215} = P(\text{Brown}')$

P(Wavy | Blond) $= \frac{5}{20}$

P(Wavy or Red) $= \frac{43}{215} + \frac{15}{215} - \frac{3}{215} = \frac{55}{215}$

$P(\text{wavy}) + P(\text{red}) - P(\text{wavy and red})$

Independence of Events

Two events are independent if the fact that one has occurred does not affect the probability of the other.

ex) • flipping 2 coins
• flipping 1 coin 10 times
• rolling 4 dice

Test for Independence

A and B are independent if any of the following is true:

1. $P(A|B) = P(A)$
2. $P(B|A) = P(B)$
3. $P(A \text{ and } B) = P(A) \cdot P(B)$

You only have to check one of these!

If A, B are not independent, we say they are dependent

Experiment: Roll a pair of dice.

O2 = orange die is a 2

D = doubles

Are O2 and D independent?

Method 1: Ck. whether $\boxed{P(O2|D) = P(O2)}$

or $P(D|O2) = P(D)$ ✓

$P(D) = \frac{6}{36} = \frac{1}{6}$

$$P(D|O2) = \frac{P(D \text{ and } O2)}{P(O2)} = \frac{1/36}{6/36} = \frac{1}{6}$$

So these are <u>independent</u> events

$P(O2) = \frac{6}{36} = \frac{1}{6}$

$$P(O2|D) = \frac{P(O2 \text{ and } D)}{P(D)} = \frac{1/36}{6/36} = \frac{1}{6}$$

Method 2:

Ck. whether

$P(D \text{ and } O2) = P(D) \cdot P(O2)$

$\boxed{P(D \text{ and } O2) = \frac{1}{36}}$

$P(D) = \frac{6}{36}$

$P(O2) = \frac{6}{36}$

$P(D) \cdot P(O2) = \frac{6}{36} \cdot \frac{6}{36} = \frac{1}{6} \cdot \frac{1}{6} = \boxed{\frac{1}{36}}$

They are independent

Conditional Probability Rule

$$P(B) \cdot P(A|B) = \frac{P(A \text{ and } B)}{\cancel{P(B)}} \cdot \cancel{P(B)}$$

Multiplication Rule: $P(A \text{ and } B) = P(A|B) \cdot P(B)$

Review of Probability Rules

- $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

- $P(\underbrace{A'}_{\text{not } A}) = 1 - P(A)$

- $P(A|B) = \dfrac{P(A \text{ and } B)}{P(B)}$

- $P(A \text{ and } B) = P(A|B) \times P(B)$

- **If A and B are independent:**

$$P(A|B) = P(A) \quad \text{and} \quad P(B|A) = P(B)$$

$$\text{and} \quad P(A \text{ and } B) = P(A) \times P(B)$$

- **If A and B are mutually exclusive:**

$$P(A \text{ and } B) = 0$$

Example: Find the following given $P(A)=0.25$, $P(B)=0.45$, and P(A | B)= 0.8.

- $P(A \text{ and } B) = P(A|B) \cdot P(B)$
  $= (0.8)(0.45)$
  $= \boxed{0.36}$

- $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
  $= 0.25 + 0.45 - 0.36$
  $= \boxed{0.34}$

- $P(A') = 1 - P(A)$
  $= 1 - 0.25 = \boxed{0.75}$

- Are A and B mutually exclusive? Why or why not? No
  $P(A \text{ and } B) = 0.36 \neq 0$

- Are A and B independent? Why or why not?
  $P(A|B) \neq P(A)$
  So they are dependent

Example:

- If P(A) = 0.60, P(A or B) = 0.85 and P(A and B) = 0.05, find P(B).

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$
$$0.85 = 0.60 + x - 0.05$$
$$\boxed{0.3 = x}$$

- If P(E and F) = 0.6 and P(E) = 0.8, what is P(F | E)?

$$P(F|E) = \frac{P(F \text{ and } E)}{P(E)}$$
$$= \frac{0.6}{0.8} = \boxed{0.75}$$

- Suppose that E and F are two events and that P(E and F) = 0.24, P(E) = 0.4 and P(F) = 0.6. Are E and F independent events? Why?

$$P(E \text{ and } F) \stackrel{?}{=} P(E) \cdot P(F) \checkmark$$

Independent

Tree Diagrams

A tree diagram is a visual way of representing the probabilities of experiments that are sequential in nature

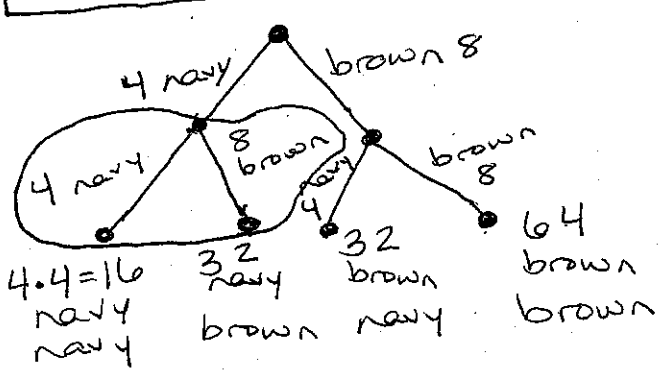
Example:

I have a sock drawer with 4 navy socks and 8 brown socks. It is a dark winter morning. I draw 2 socks at random from the drawer.

Two ways to do this:

- with replacement
- without replacement

with replacement



Total: $16+32+32+64=144$

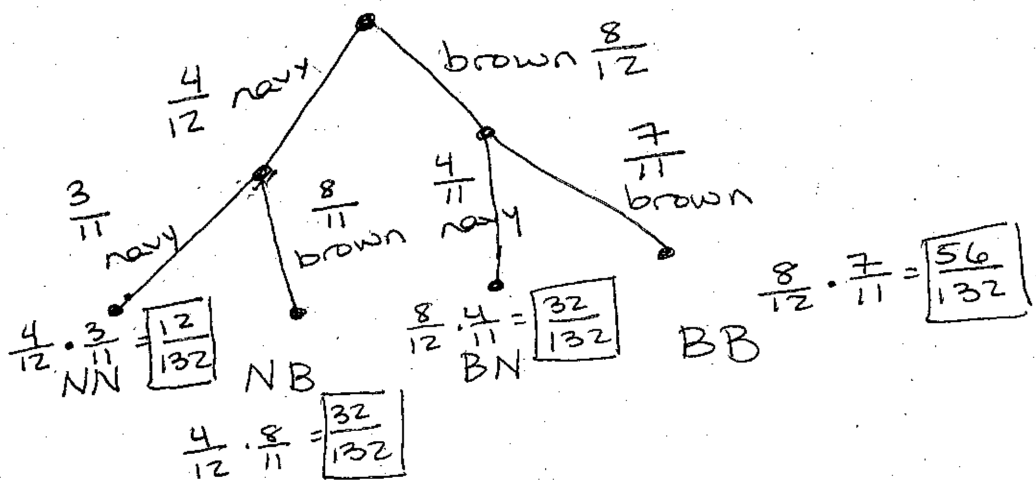
With replacement:

What is the probability of drawing:

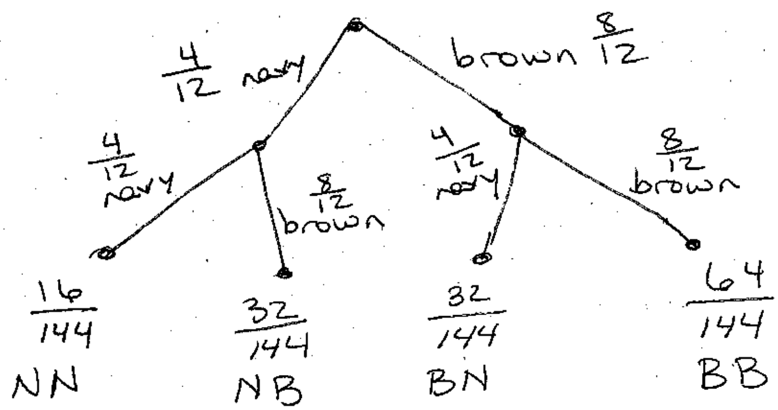
- two Brown socks? $\frac{64}{144}$
- mismatched socks? $\frac{32+32}{144} = \frac{64}{144}$
- a navy sock on the 2nd draw? $\frac{16+32}{144} = \frac{48}{144}$
- another navy sock if you already drew a navy sock on the 1st draw?

  $P(\text{navy } 2^{nd} \mid \text{navy } 1^{st}) = \frac{4}{12}$

Without replacement



With replacement, revisited



Without replacement:

What is the probability of drawing:

- two Brown socks? $\frac{56}{132}$

- mismatched socks? $\frac{32}{132} + \frac{32}{132} = \frac{64}{132}$

$$P(BN \text{ or } NB) = P(BN) + P(NB) - P(BN \text{ and } NB)$$
$$= \frac{32}{132} + \frac{32}{132} - 0$$

- a navy sock on the 2nd draw?

$$P(BN \text{ or } NN) = \frac{12}{132} + \frac{32}{132} = \frac{44}{132}$$

- another navy sock if you already drew a navy sock on the 1st draw? $\boxed{\frac{3}{11}}$

$$P(NN \mid NN \text{ or } NB) = \frac{P(NN)}{P(NN \text{ or } NB)} = \frac{12/132}{44/132}$$
$$= \frac{12 \div 4}{44 \div 4} = \frac{3}{11}$$