Notation conventions in this lab:

- $f^{(n)}(x)$ is the $n^{\text{th}}$ derivative of $f(x)$
- $T_n(x)$ is an $n^{\text{th}}$ degree Taylor polynomial
- $R_n(x) = f(x) - T_n(x)$ is the remainder of an $n^{\text{th}}$ degree Taylor polynomial

Some functions are nicer than others. Who wouldn't rather work with polynomials than some complex composition of logarithmic and trigonometric functions? This is the power of Taylor polynomials. However, in order to work with a polynomial rather than some complex function, we need to be aware of how closely the polynomial approximates the function. This will require us to **bound the error** for our approximations. Before we can bound the error on our approximations we will need to know how to place bounds on a function.

**I.** a) Find the maximum of $x^3 - 3x^2 - 3x - 4$ on the interval $[0,2]$.

b) Suppose $f^{(n+1)}(x) = x^3 - 3x^2 - 3x - 4$. Explain why your answer in part (a) is not enough to help you find a bound on $R_n(x)$.

c) Using the same $f^{(n+1)}(x)$ as in (b), find $M$ such that $\left|f^{(n+1)}(x)\right| \leq M$ on the interval $[0,2]$.

**II.** We can often use the **triangle inequality** $|f(x) + g(x)| \leq |f(x)| + |g(x)|$ to help us find bounds on functions.

a) Use the triangle inequality together with the fact that for any numbers $a$ and $b$, $|ab| = |a| \cdot |b|$ to clearly show that $|f(x) - g(x)| \leq |f(x)| + |g(x)|$.

b) **Use the triangle inequality** to find bounds for $f(x) = x^3 - 7x^2 + 3x - 5$ on the interval $[-2, 1]$ by breaking $f(x)$ into four monomials. Be careful to justify all of your steps.

c) Let $f(x) = x^2 \sin x - 3x \cos x$. Find a bound on $|f(x)|$ on the interval $[0, \pi]$ **using the techniques from II.a and II.b**. Be careful to clearly justify all of your steps.

**III.** Consider the function $f(x) = 4x^2 e^{-x} - 8xe^{-x} + e^{-x}$ on the interval $[0, 2]$.

a) Make a careful **triangle inequality argument** to find a bound on $|f(x)|$.

b) We can view $f(x)$ as the product of the functions $g(x) = 4x^2 - 8x + 1$ and $h(x) = e^{-x}$. Find the maximums of $|g(x)|$ and $|h(x)|$ on the interval $[0, 2]$. Use these maximums to come up with a different bound on $|f(x)|$ on the interval $[0, 2]$.

c) Use the methods of Math 1A to find the actual maximum of $|f(x)|$ on the interval $[0, 2]$.

d) You have now found three different values for $M$ such that $|f(x)| \leq M$ on the interval $[0, 2]$. For each of these values, write a sentence or two describing (i) the accuracy of the value and (ii) how easy the value was to calculate. Then decide which technique you think is "best" for this function by considering (i) and (ii), as well as which value of M you would prefer to work with.

Now that we have practiced finding bounds on functions, we can actually get our hands dirty approximating things. In each of the following problems, you will need an **error bound** $|R_n(x)|$. In order to have an error bound, you will need to find $M$ such that $\left|f^{(n+1)}(x)\right| \leq M$. In each case, it is not enough to find $M$; you need to make sure you **include an explanation** of the methods you used and why that particular choice of $M$ works.

**IV.** Consider $f(x) = \frac{1}{x}$.

a) Find the 5th degree Taylor polynomial for $f(x)$ centered at $-2$.

b) Find an error bound for $T_5(x)$ on the interval $[-3, -1]$.

**V.** Consider $f(x) = \sqrt[3]{x}$.

a) Use the 3rd degree Taylor polynomial for $f(x)$ centered at 8 to approximate $\sqrt[3]{8.5}$.

b) Find an error bound for the approximation you came up with in (a).

**VI.** Use a Taylor polynomial centered at $\frac{\pi}{2}$ to approximate $\sin(89°)$ to within $0.001$ of its actual value.

**VII.** Find a polynomial approximation for $f(x) = x \sin x$ on the interval $[-\pi, \pi]$ that is accurate to within 0.01.

**VIII.** Consider $f(x) = e^{-x^2}$.

a) Find the 3rd degree Taylor polynomial for $f(x)$ centered at 0.

b) Find an error bound for $T_3(x)$ when $|x| \leq 1$.

**IX.** Consider $f(x) = \sin(x^2)$.

a) Find the 2nd degree Taylor polynomial for $f(x)$ centered at 0.

b) Find a bound on the error if $T_2(x)$ is used to approximate $\sin\left(\frac{\pi^2}{36}\right)$.

**X.** Finally, we'll look at a function that does not agree with its Maclaurin series.

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

a) Find $f'(x)$ for $x \neq 0$.

b) Use the **definition of the derivative** to find $f'(0)$.

c) Use the **definition of the derivative** to find $f''(0)$.

d) Use the results of (a) and (b) to make a conjecture about $f^{(n)}(0)$. Based on this conjecture what is $T_n(x)$?

e) Find the maximum error for the approximation $T_n(x)$ on any interval $|x| \leq d$. Conclude that the function only agrees with its Maclaurin series at the origin.