Ibn–Alhaitham Begets Gauss

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Carl Friedrich Gauss contributions to Physics and Mathematics are well know to students of science and Mathematics, and one of his early childhood discoveries told in different interesting ways is the sum of the first n positive integers.

$$1+2+3+4+\ldots\ldots+n=\frac{n}{2}(n+1)$$

Using summation notation we get

★ $$\sum_{i=1}^{n} i=\frac{n}{2}(n+1)$$

for example if we take n = 10, then

$$\begin{aligned}&1+2+3+4+5+6+7+8+9+10\\&=(1+10)+(2+9)+(3+9)+(4+7)+(5+6)\\&=11+11+11+11+11\\&=5(11)\\&=\frac{10}{2}(1+10)\end{aligned}$$

Now imagine that we can do this for any positive integer n, say n = one million and find the sum by multiplying half a million by one million and one to sum up the integers from 1 to million.

The general result ★ can be proven geometrically, shown in the figure below, or by principle of mathematical induction.

Gauss' formula is a special case of a much more difficult and interesting result discovered about 1000 earlier in Alexandra by Al–Hassan Ibn–Alhaitham.

★^{k} $$(n+1)\sum_{i=1}^{n} i^k=\sum_{i=1}^{n} i^{k+1}+\sum_{i=1}^{n}\left(\sum_{j=1}^{i} j^k\right)$$

This formula can be used to find "quickly" sums of powers of the kth powers of the first n positive integers:

$1^k+2^k+3^k+4^k+\ldots..+n^k$ for any positive integers n and k.

As an example, we can use Al-Hassan's formula to derive a formula for the sum of squares of the first n integers:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

and specifically, for n =10, the above formula gives

$$1+4+9+16+25+36+49+64+81+100 = \frac{10(10+1)(2(10)+1)}{6} = 385$$

which can always be computed easily by an addition, two multiplication and division by 6.

Now if we consider $\bigstar^0$, which is $\bigstar^k$ with k = 0.

$$(n+1)\sum_{i=1}^{n} i^0 = \sum_{i=1}^{n} i^{0+1} + \sum_{i=1}^{n}\left(\sum_{j=1}^{i} j^0\right)$$

$$(n+1)\sum_{i=1}^{n} 1 = \sum_{i=1}^{n} i + \sum_{i=1}^{n}\left(\sum_{j=1}^{i} 1\right)$$

$$(n+1)n = \sum_{i=1}^{n} i + \sum_{i=1}^{n}(i)$$

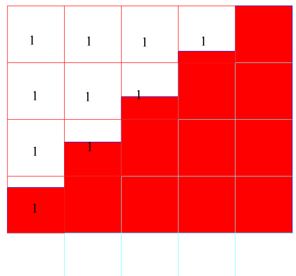
$$(n+1)n = 2\sum_{i=1}^{n} i$$

$$\frac{(n+1)n}{2} = \sum_{i=1}^{n} i$$

And we conclude that Ibn–Alhaitham begets Gauss.

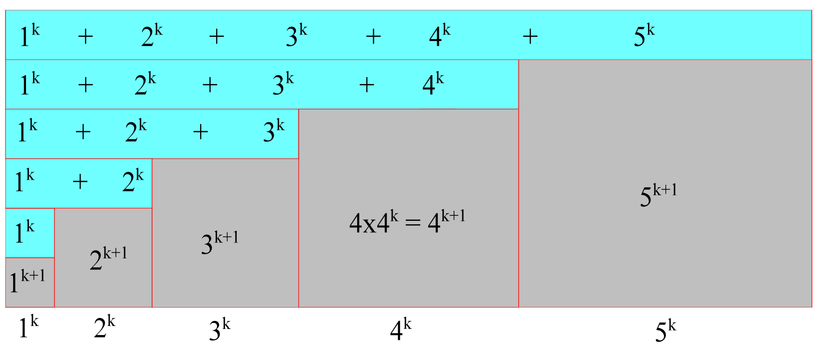
The geometric proof for Gauss's formula:

The figure below illustrate the geometric proof by adding the first four integers, n = 4, and can be extended to any integer n. The area of the white squares which is the sum of the first four positive integers is half the area of the rectangle which is $4(4+1)$. This is the right side of Gauss' formula (n/2)(n+1)



Al–Hassan's Formula requires more sophisticated geometric proof, but still uses the idea of area representing sums of positive integers or sums of any positive powers of integers.

The figure below is divided into rows in teal and columns in gray. These rows and columns both represent areas of rectangles. The rows have heights equal unity and their widths are sums of powers of positive integers. The top row shows the sum of k^{th} powers of the first five positive integers. The total area for each row is the sum of the numbers written inside it.
The columns have area equal to the k^{th} power of the integers from 1 to 5. The figure can be extended to any number n of columns which will create n+1 rows.



In $★^k$, the left hand side represents the total area of the rectangle, which is the height of n +1 units times the width which is the sum of the widths of the vertical rectangles.

ie Total area $= (n+1)\sum_{i=1}^{n} i^k$

The right side of the formula contains the sum of the areas of the gray columns and the sum of the sums of the teal rows.

That is the total area $= \sum_{i=1}^{n} i^{k+1} + \sum_{i=1}^{n}\left(\sum_{j=1}^{i} j^k\right)$.