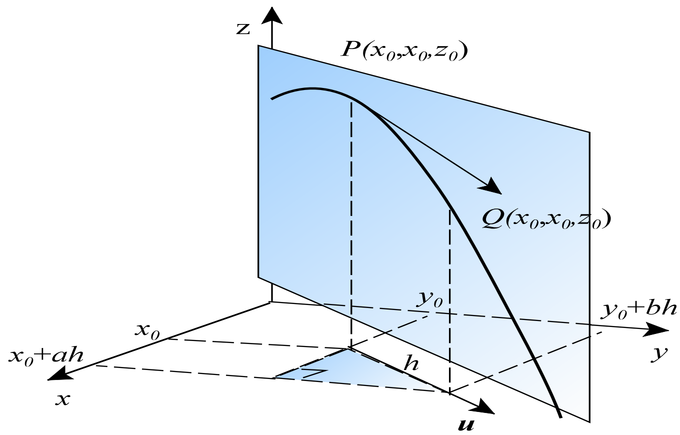

Notation:

$D_{\mathbf{u}} f(x, y)$ The instantaneous rate of change of f in the direction of **u**.

So, $D_{\mathbf{i}} f(x, y) = \dfrac{\partial f}{\partial x}$ and $D_{\mathbf{j}} f(x, y) = \dfrac{\partial f}{\partial y}$

Let the curve C be the trace of the surface $z = f(x, y)$ on a vertical plane parallel to the unit vector **u**, and the points $P(x_0, y_0, z_0)$ and $Q(x_0 + ah, y_0 + bh, z_0)$ are on C.

Suppose $z = f(x, y)$ has continuous first partial derivatives.

Define
$$D_{\mathbf{u}} f(x, y) = \underset{h \to 0}{Lim} \frac{z - z_0}{h} = \underset{h \to 0}{Lim} \frac{\Delta z}{h}$$
$$= \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

$$\text{Let } g(h) = f(x_0 + ah, y_0 + bh) = f(x, y)$$

$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = D_{\mathbf{u}} f(x_0, y_0) \quad [1]$$

$$g'(h) = f_x(x, y)a + f_y(x, y)b \qquad [2]$$

$$\text{Take } h = 0, \ g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$$

Using [1] and [2], we get

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$$

$$D_{\mathbf{u}} f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \bullet \langle a, b \rangle$$

The Gradient

Definition

Suppose $z = f(x, y)$ has continuous first partial derivatives, the Gradient of f is defined to be

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

Using the gradient, we can write the directional derivative as

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u}$$

Remark

Note that the maximum rate of change of f occurs when the unit vector u is parallel to the gradient.

Consider the level surface F πt on the level Surface $\mathrm{F}(x,y,z) = k$

We have $\nabla F(x,y,z) = \langle F_x, F_y, F_z \rangle$

And $\mathbf{r}'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$

We also have $\frac{dF}{dt} = \left\langle F_x \frac{dx}{dt}, F_y \frac{dy}{dt}, F_z \frac{dz}{dt} \right\rangle = \langle F_x, F_y, F_z \rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle,$

for every point $\mathrm{P}(x,y,z)$ on C.

Note that at $\mathrm{P}_0(x_0, y_0, z_0)$ $\frac{dF}{dt}$ =0, so ∇F it is perpendicular to the tangent vector $\mathbf{r}'(t_0)$.

In other words, the gradient to the level surface $\mathrm{F}(x,y,z) = k$ at the point (x_0, y_0, z_0) is orthagonal to the trace C defined by $\mathbf{r}(\mathrm{t})$ at the point $\mathbf{r}(\mathrm{t}_0) = \langle x_0, y_0, z_0 \rangle$.

This fact can be extended to every curve $\mathbf{r}(\mathrm{t})$ that passes through $P(x_0, y_0, z_0)$.
Hence ∇F is perpendicular to all curves $\mathbf{r} = \mathbf{r}(\mathrm{t})$ that contain the point (x_0, y_0, z_0), which in turn implies that $\nabla F(x_0, y_0, z_0)$ is orthagonal to the surfac $\mathrm{F}(x,y,z) = k$, at (x_0, y_0, z_0).

This conclude that the gradient to a level surface at any point $P(x,y,z)$ is perpendicular to the tangent plane there.

ie $\nabla F(x,y,z) \perp \mathrm{F}(x,y,z) = k$