Extreme Values of functions H. Bourgoub

The Method of Lagrange Multipliers

The problem is to find the maximum or minimum values of $z = f(x, y)$, when the domain of *f* is restricted by $g(x, y) = c$.

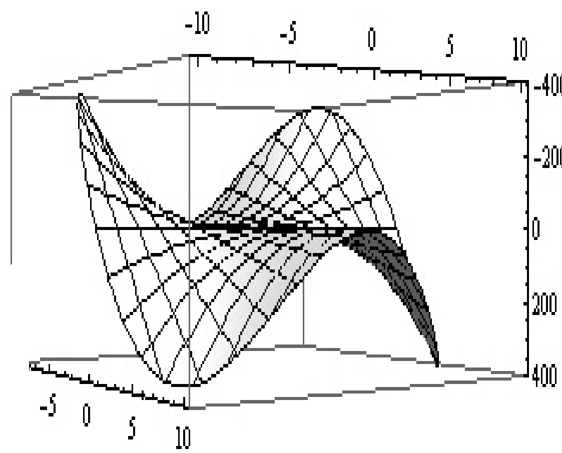
The Method

If we march along paths on the xy plane t hat lie on the curve $g(x, y) = c$ we will be marching among contours of $z = f(x, y)$. There, we either intersect these contour curves of *f* or come tangential to them. If we cross contour lines of *f* , then the values of *f* on these lines either continue to increase or decrease or remain constant. But in the event that we come tangent to a contour curve
of *f* at a point (*x* , *y*), it means either we reached a minimum or a maximum value of *f* and returning to larger or smaller values respectively. This indicate that extreme values of *f* occur at points where the constraint,
$g(x, y) = c$ and the level curves $f(x, y) = c$ are tangential, or have the same tangent line. This implies that the gradients of *f* and *g* are parallel.

So, to find extreme values of $z = f(x, y)$ subject to $g(x, y) = c$ is to find all points (*x* , *y*)

where, $\nabla f(x, y) = \lambda \nabla g(x, y)$, and $g(x, y) = c$ with $\lambda \neq 0$, and $\nabla g(x, y) \neq \langle 0, 0 \rangle$ then

compare the values of *f* at these points to establish the extreme values.



Example
Find the extreme values
of $f(x, y) = xy^2$, subject to $x^2 + 2y^2 = 6$.

$$\begin{cases} \langle y^2, 2xy \rangle = \lambda \langle 2x, 4y \rangle \\ x^2 + 2y^2 = 6 \end{cases}$$

$$\begin{cases} y^2 = 2\lambda x & (1) \\ 2xy = 4\lambda y & (2) \\ x^2 + 2y^2 = 6 & (3) \end{cases}$$

From equations 2, we get

$\lambda = x/2$

And substituting λ in equation 3, we get

$3y^2 = 6$

$y = \pm\sqrt{2}$, then substituting x and y into the constraint

$x = \pm\sqrt{2}$, $\lambda = \pm\sqrt{2}/2$ and

The maximum value of f is $f(\pm 2, 1) = 2\sqrt{2}$

The minimum value of f is $f(\pm 2, -1) = -2\sqrt{2}$