VECTORS

DEF: A vector is a quantity that has both magnitude and direction.

DEF: A scalar is a quantity that has magnitude but NO direction.

Ex.	Vectors	Ex.	Scalars
	Force		Temperature
	Velocity		Time
	Displacement		Mass
	Momentum		Speed

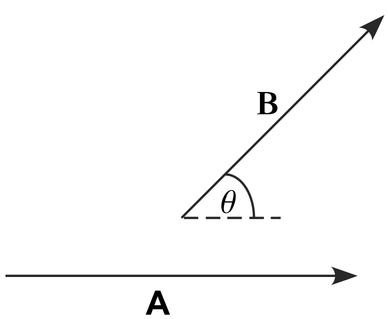
Vector Notation

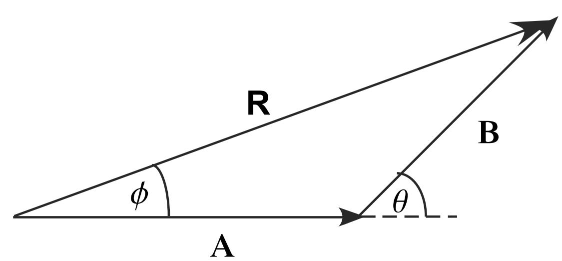
A – Boldface letters
$\vec{A}$ - Arrow above letter
A – Magnitude of vector **A**

A vector is defined graphically by an arrow whose length is proportional to the magnitude of the vector quantity. The direction of the arrow points in the direction of the vector quantity.

Adding Vectors Graphically

Consider adding two vectors A and B graphically. The two vectors are shown below.





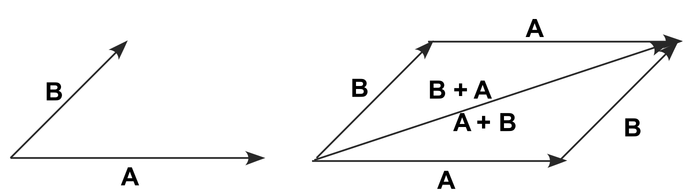
1. Select an appropriate scale. (Ex. 20 cm = 5 N)
2. Draw vector **A** to scale and in the proper direction.
3. Draw vector **B** to the same scale with its tail at the tip of **A** and in the proper direction.
4. The resultant vector **R** = **A** + **B** is the vector drawn from the tail of vector **A** to the tip of vector **B**.
5. Calculate the magnitude of the resultant vector **R** using the selected scale and measure its direction with a protractor.
6. This same process applies if you add more than two vectors.

This method of adding vectors graphically is also referred to as the head-to-tail method, analytical method, and geometric method.

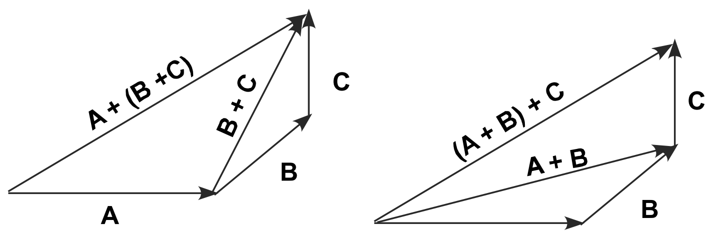
Properties of Vectors

1. **A** + **B** = **B** + **A** (commutative Law)
2. **A** + **(B** + **C)** = **(A** + **B)** + **C** (associative Law)
3. **A** + **(-A)** = **0** (Negative of a vector)
4. **A** – **B** = **A** + **(-B)** (vector subtraction)
5. **B** = s**A** (vector multiplied by a scalar)

(1)

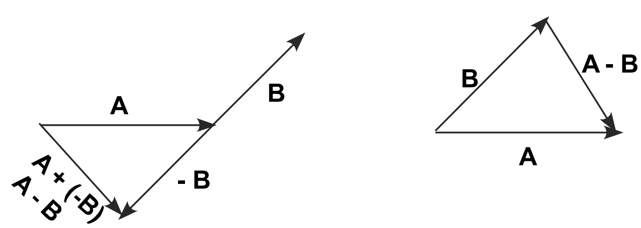


(2)



(3)

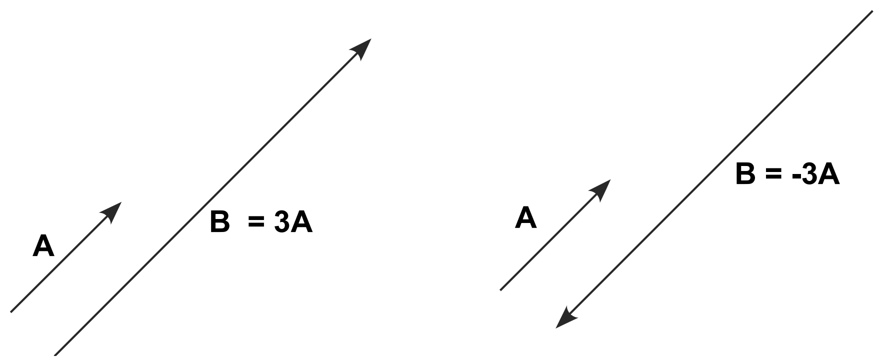
(4)



(5) **B** = s**A** is a vector with magnitude $|s|$A and is parallel to **A** if s is positive and antiparallel if s is negative.

Ex. s = 3
B = 3**A**

Ex. s = 3
B = -3**A**



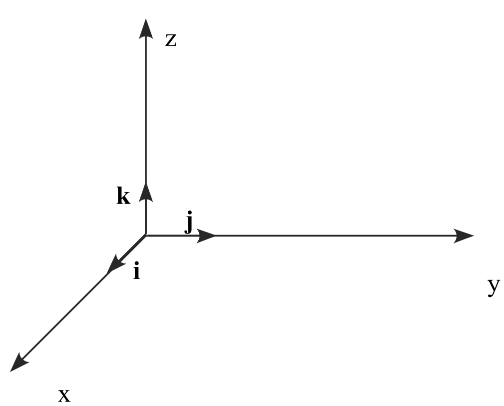
Unit Vectors

Def: A unit vector is a dimensionless vector, one unit in length, used to specify a given direction in space.

- They have no physical significance and are only used to specify a given direction in space.

Notation:

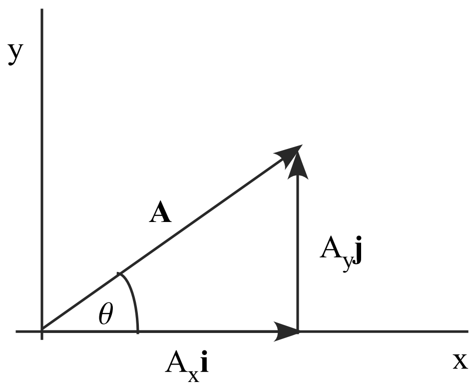
$\hat{i}$ = unit vector pointing along the positive x – axis
$\hat{j}$ = unit vector pointing along the positive y – axis
$\hat{k}$ = unit vector pointing along the positive z – axis



Any vector can be expressed in terms of unit vectors.

Components of A Vector

Consider a two dimensional vector **A**. The vector can be written in unit vector notation.



A = A_x**i** + A_y**j**

$A_x = A \cos \theta$
$A_y = A \sin \theta$

- A_x and A_y are not vectors. They are scalar quantities that can be positive or negative depending on the angle θ. A_x and A_y are called the scalar components of vector **A**.

$|\mathbf{A}| = A = \sqrt{A_x^2 + A_y^2}$ Magnitude of vector **A**

$$\tan\theta = \frac{A_y}{A_x}$$

$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$ Direction of **A** relative to positive x-axis

Adding Vectors Using Component Method

Consider adding three 2-D vectors **A**, **B**, and **C**:

A = A_x**i** + A_y**j**
B = B_x**i** + B_y**j**
C = C_x**i** + C_y**j**

1. Add the x-components and y-components of each vector to obtain the resultant vector **R** in unit vector notation.

   R = **A** + **B** + **C** = (A_x**i** + A_y**j**) + (B_x**i** + B_y**j**) + (C_x**i** + C_y**j**)

   R = (A_x + B_x + C_x) **i** + (A_y + B_y + C_y) **j**

   $$R_x = A_x + B_x + C_x$$

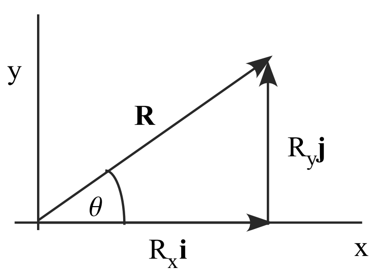
   $$R_y = A_y + B_y + C_y$$

   $$\mathbf{R} = R_x\,\mathbf{i} + R_y\,\mathbf{j}$$

2. Calculate the magnitude of the resultant vector **R**.

   $$R = \sqrt{R_x^2 + R_y^2}$$

3. Calculate the direction of **R** relative to the positive x – axis.



$$\tan\,\theta = \frac{R_y}{R_x}$$

$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$

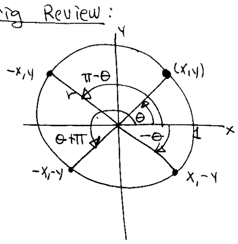
4. Same procedure applies if you add more than 3 vectors. However, if the vectors are 3D, then you must specify the direction of the resultant vector **R** relative to the positive x, y, and z axis.

Note:

R_x **i** and R_y **j** are called the vector components of vector **R**. R_x and R_y are called the scalar components or simply components of **R.** R_x and R_y are scalar quantities which can be zero, positive, or negative. It is a pure number with no direction.

Suppose the components (scalar) of a vectors **R** are -2 and 3. These scalar numbers do not tell you the direction of the vector **R**! If you want to specify the direction of **R** you need to use unit vector: **R** = -2**i** + 3**j**

Trig Review:



$\cos(\pi \pm \theta) = -\cos\theta$

$\sin(\pi \pm \theta) = \mp \sin\theta$

$\tan\theta = \tan(\theta + \pi)$

$\tan\theta = -\tan(\pi - \theta) = \tan(\theta - \pi)$

$\therefore \tan\theta = \tan(\theta \pm \pi)$

$\tan\theta = \tan(\theta \pm n\pi) \quad n = 0, 1, 2, \ldots$

$\sin(\pi - \theta) = \frac{y}{r} = \sin\theta \Rightarrow \sin(\pi-\theta) = \sin\theta$

$\cos(\pi - \theta) = \frac{-x}{r} = -\cos\theta \Rightarrow \cos(\pi-\theta) = -\cos\theta$

$\tan(\pi - \theta) = -\tan\theta$

$\sin(\pi + \theta) = \frac{-y}{r} = -\sin\theta \Rightarrow \sin(\pi+\theta) = -\sin\theta$

$\cos(\pi + \theta) = \frac{-x}{r} = -\cos\theta \Rightarrow \cos(\pi+\theta) = -\cos\theta$

$\tan(\pi + \theta) = \tan\theta$

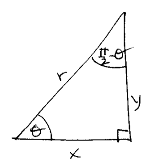
$\sin\theta = \frac{y}{r} = -\sin(-\theta) \Rightarrow \sin\theta = -\sin(-\theta)$

$\cos\theta = \frac{x}{r} = \cos(-\theta) \Rightarrow \cos\theta = \cos(-\theta)$

$\tan\theta = -\tan(-\theta)$

$\sin\theta = \sin(\theta \pm 2\pi n) \quad n = 0, 1, 2, \ldots$

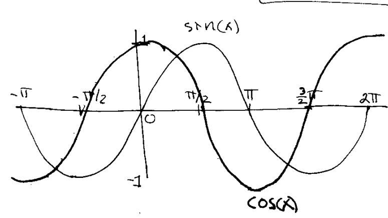
$\cos\theta = \cos(\theta \pm 2\pi n) \quad n = 0, 1, 2, \ldots$



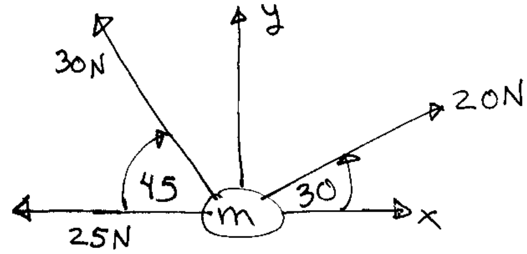
$\sin\theta = \frac{y}{r} = \cos(\frac{\pi}{2} - \theta) \Rightarrow \sin\theta = \cos(\frac{\pi}{2} - \theta)$

$\cos\theta = \frac{x}{r} = \sin(\frac{\pi}{2} - \theta) \Rightarrow \cos\theta = \sin(\frac{\pi}{2} - \theta)$

$\tan\theta = \text{ctn}(\frac{\pi}{2} - \theta)$



Ex. Consider 3 forces acting on a particle. Determine the resultant force on the particle.



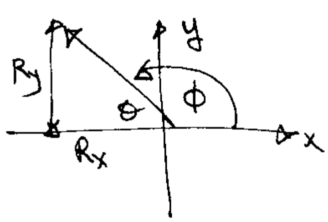
x-comp	y-comp
$20\cos 30$	$20\sin 30$
$-30\cos 45$	$30\sin 45$
-25	0
$R_x = -28.9\text{ N}$	$R_y = 31.2\text{ N}$

$$\vec{R} = R_x\hat{i} + R_y\hat{j}$$

$$\vec{R} = -28.9\hat{i} + 31.2\hat{j}$$

$$R = \sqrt{R_x^2 + R_y^2}$$

$$R = \sqrt{(-28.9)^2 + (31.2)^2} = 42.5\text{ N}$$



$$\tan\theta = \frac{R_y}{R_x} = \frac{31.2}{28.9}$$

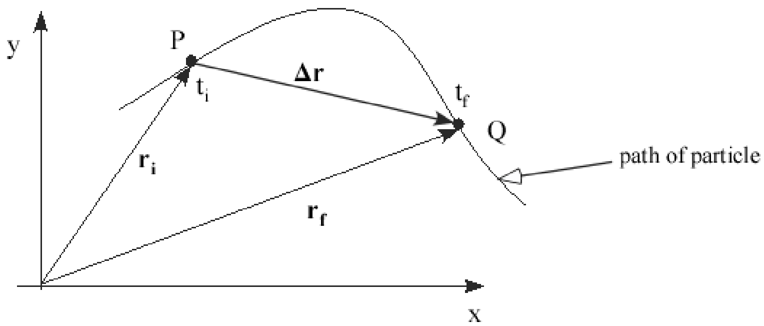
$$\theta = \tan^{-1}\left(\frac{31.2}{28.9}\right)$$

$$\theta = 47.2^\circ$$

$$\phi = 180 - 47.2 = 132.8^\circ$$

Velocity Vector

Recall that velocity is a vector with both magnitude and direction. Consider the motion of a particle between two points P and Q.



$\boxed{\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}$ Position Vector

$$\boxed{\begin{aligned}\mathbf{r_i} &= x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k}\\ \mathbf{r_f} &= x_f\mathbf{i} + y_f\mathbf{j} + z_f\mathbf{k}\end{aligned}}$$

$$\mathbf{r}_i + \Delta\mathbf{r} = \mathbf{r_f}$$

$\boxed{\Delta\mathbf{r} = \mathbf{r_f} - \mathbf{r}_i}$ Displacement Vector

$\boxed{\mathbf{V_{ave}} = \frac{\Delta\mathbf{r}}{\Delta t} = \frac{\mathbf{r_f} - \mathbf{r_i}}{t_f - t_i}}$ Average Velocity Vector

$$\Delta\mathbf{r} = \mathbf{r_f} - \mathbf{r_i} = (x_f - x_i)\mathbf{i} + (y_f - y_i)\mathbf{i} + (z_f - z_i)\mathbf{i}$$
$$\Delta\mathbf{r} = \Delta x\mathbf{i} + \Delta x\mathbf{j} + \Delta x\mathbf{k}$$

$\boxed{\mathbf{v_{ave}} = \frac{\Delta\mathbf{r}}{\mathbf{\Delta}t} = \left(\frac{\Delta x}{\mathbf{\Delta}t}\right)\mathbf{i} + \left(\frac{\Delta y}{\mathbf{\Delta}t}\right)\mathbf{j} + \left(\frac{\Delta x}{\mathbf{\Delta}t}\right)\mathbf{k}}$ Average Velocity Vector

To obtain the instantaneous velocity vector we calculate the average velocity over smaller and smaller time intervals Δt. In the limit as $\Delta t \to 0$, the average velocity vector will approach the instantaneous velocity vector.

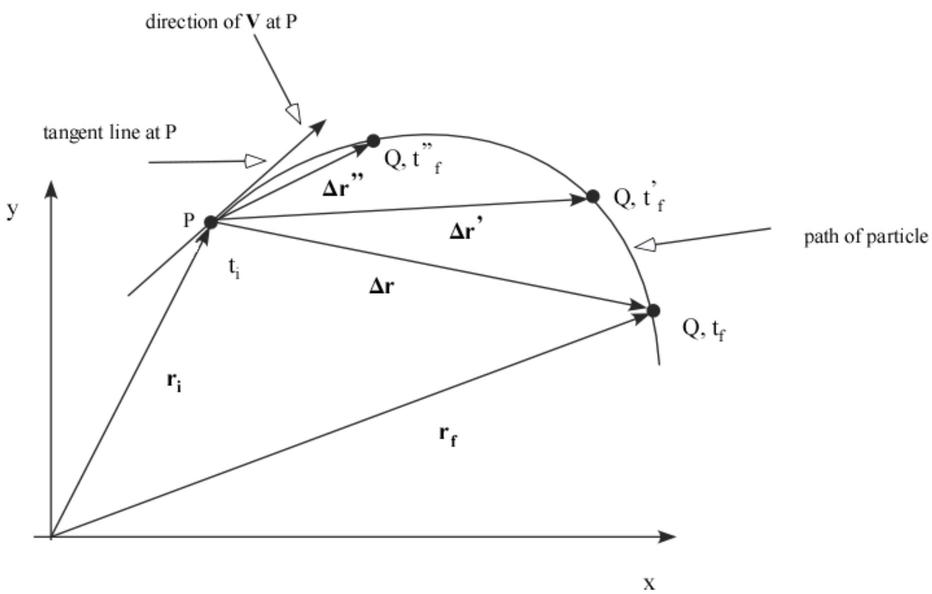
$\boxed{\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{r}}{\mathbf{\Delta t}} = \frac{d\mathbf{r}}{dt}}$ Instantaneous Velocity

Since $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right) = \left(\frac{dx}{dt}\right)\mathbf{i} + \left(\frac{dy}{dt}\right)\mathbf{j} + \left(\frac{dz}{dt}\right)\mathbf{k}$$

$$\boxed{\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}}$$ Velocity Vector

Now how do we determine the direction of the velocity vector as the particle moves along the path?? If you look at the definition of the velocity vector $\boxed{\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}}$ you can see that the direction of **v** is in the same direction as **Δr** in the limit as Δt → 0. As an example let's determine the direction of **v** at point P. Let's firsts draw a tangent line to the curve at point P and analyze the direction of **Δr** between point P and point Q in the limit as Δt → 0. As Δt gets smaller and smaller you can see that point Q gets closer and closer to point P along the path and **Δr** begins to approach the tangent line at P. In the limit as Δt → 0, we can see that **Δr** becomes tangent to point P and directed in the direction of motion.

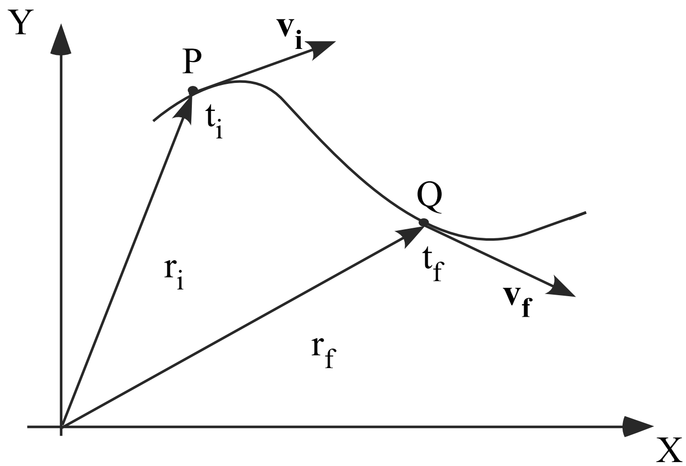


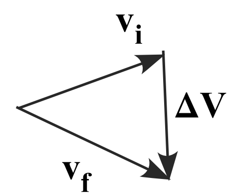
1. The direction of **v** is tangent to the path of the particle and in the direction of motion.

2. $|\mathbf{v}| = speed$

Acceleration Vector

Again consider a particle moving between two points P and Q.





$\boxed{\Delta \mathbf{v} = \mathbf{v}_f - \mathbf{v}_i}$ Change in Velocity

$\boxed{a_{ave} = \dfrac{\Delta \mathbf{v}}{\Delta t} = \dfrac{\mathbf{v}_f - \mathbf{v}_i}{t_f - t_i}}$ Average Acceleration

$$\mathbf{v}_i = v_{xi}\mathbf{i} + v_{yi}\mathbf{j} + v_{zi}\mathbf{k}$$
$$\mathbf{v}_f = v_{xf}\mathbf{i} + v_{yf}\mathbf{j} + v_{zf}\mathbf{k}$$
$$\Delta \mathbf{v} = (v_{xf} - v_{xi})\mathbf{i} + (v_{yf} - v_{yi})\mathbf{j} + (v_{zf} - v_{zi})\mathbf{k}$$
$$\Delta \mathbf{v} = \Delta v_x \mathbf{i} + \Delta v_y \mathbf{j} + \Delta v_z \mathbf{k}$$

$\boxed{\mathbf{a}_{ave} = \dfrac{\Delta \mathbf{v}}{\Delta t} = \left(\dfrac{\Delta v_x}{\Delta t}\right)\mathbf{i} + \left(\dfrac{\Delta v_y}{\Delta t}\right)\mathbf{y} + \left(\dfrac{\Delta v_z}{\Delta t}\right)\mathbf{k}}$ Average Acceleration

$\boxed{\mathbf{a} = \lim_{\Delta t \to 0} \dfrac{\Delta \mathbf{v}}{\Delta t} = \dfrac{d\mathbf{v}}{dt}}$ Instantaneous Acceleration

Since $\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}$, then

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}\right) = \left(\frac{dv_x}{dt}\right)\mathbf{i} + \left(\frac{dv_y}{dt}\right)\mathbf{j} + \left(\frac{dv_z}{dt}\right)\mathbf{k}$$

$\boxed{\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}}$ Instantaneous Acceleration

A change in the velocity vector will always cause an acceleration. This will occur when:

1. the magnitude of **v** changes (linear motion)
2. the direction of **v** changes (circular motion)
3. both the direction and magnitude of **v** change