Chapter 1
Sentential Logic

1.1 Basic logical notions

argument, premises, conclusion

Definition. An **ARGUMENT** is a pair of things:
- a set of sentences, the **PREMISES**
- a sentence, the **CONCLUSION**.

Comment. All arguments have conclusions, but not all arguments have premises: the set of premises can be the empty set! Later we shall examine this idea in some detail.

Comment. If the sentences involved belong to English (or any other natural language), we need to specify that the premises and the conclusion are sentences that can be true or false. That is, the premises and the conclusion must all be declarative (or indicative) sentences such as 'The cat is on the mat' or 'I am here', and not sentences such as 'Is the cat on the mat?' (interrogative) or 'Come here!' (imperative). We are going to construct some formal languages in which every sentence is either true or false. Thus this qualification is not present in the definition above.

validity

Definition. An argument is **VALID** if and only if it is necessary that *if* all its premises are true, its conclusion is true.

Comment. The intuitive idea captured by this definition is this: If it is possible for the conclusion of an argument to be false when its premises are all true, then the argument is not reliable (that is, it is invalid).

If true premises guarantee a true conclusion then the argument is valid.

Alternate formulation of the definition. An argument is **VALID** if and only if it is impossible for all the premises to be true while the conclusion is false.

entailment

Definition. When an argument is valid we say that its premises **ENTAIL** its conclusion.

soundness

Definition. An argument is **SOUND** if and only if it is valid and all its premises are true.

Comment. It follows that all sound arguments have true conclusions.

Comment. An argument may be unsound in either of two ways: it is invalid, or it has one or more false premises.

Disc. Q: Why? –

Comment. The rest of this book is concerned with validity rather than soundness.

Exercise 1.1

Indicate whether each of the following sentences is True or False.

i* Every premise of a valid argument is true.
ii* Every invalid argument has a false conclusion.
iii* Every valid argument has exactly two premises.
iv* Some valid arguments have false conclusions.
v* Some valid arguments have a false conclusion despite having premises that are all true.

vi* A sound argument cannot have a false conclusion.
vii* Some sound arguments are invalid.
viii* Some unsound arguments have true premises.
ix* Premises of sound arguments entail their conclusions.
x* If an argument has true premises and a true conclusion then it is sound.

1.2 A Formal Language for Sentential Logic

formal language *Comment.* To represent similarities among arguments of a natural language, logicians introduce formal languages. The first formal language we will introduce is the language of sentential logic (also known as propositional logic). In chapter 3 we introduce a more sophisticated language: that of predicate logic.

vocabulary *Definition.* The **VOCABULARY OF SENTENTIAL LOGIC** consists of

- **SENTENCE LETTERS**,
- **CONNECTIVES**, and
- **PARENTHESES.**

sentence letter *Definition.* A **SENTENCE LETTER** is any symbol from the following list:

$$A, \dots, Z, A_0, \dots, Z_0, A_1, \dots, Z_1, \dots .$$

sentence variable *Comment.* By the use of subscripts we make available an infinite number of sentence letters. These sentence letters are also sometimes called **SENTENCE VARIABLES**, because we use them to stand for sentences of natural languages.

connectives *Definition.* The **SENTENTIAL CONNECTIVES** (often just called **CONNECTIVES**) are the members of the following list: ~, &, ∨, →, ↔.

Comment. The sentential connectives correspond to various words in natural languages that serve to connect declarative sentences.

tilde ~ The **TILDE** corresponds to the English ‘It is not the case that’. (In this case the use of the term ‘connective’ is odd, since only one declarative sentence is negated at a time.)

ampersand & The **AMPERSAND** corresponds to the English ‘Both … and …’.

wedge ∨ The **WEDGE** corresponds to the English ‘Either … or …’ in its inclusive sense.

arrow → The **ARROW** corresponds to the English ‘If … then …’.

double-arrow ↔ The **DOUBLE-ARROW** corresponds to the English ‘if and only if’.

Comment. Natural languages typically provide more than one way to express a given connection between sentences. For instance, the sentence 'John is dancing but Mary is sitting down' expresses the same logical relationship as 'John is dancing and Mary is sitting down'. The issue of translation from English to the formal language is taken up in section 1.3.

) and (The right and left parentheses are used as punctuation marks for the language.

expression *Definition.* An **EXPRESSION** of sentential logic is any sequence of sentence letters, sentential connectives, or left and right parentheses.

Examples.
(P → Q) is an expression of sentential logic.
)PQ→~ is also an expression of sentential logic.
(3 → 4) is not an expression of sentential logic.

metavariable *Definition.* Greek letters such as ϕ and ψ are used as **METAVARIABLES.** They are not themselves parts of the language of sentential logic, but they stand for expressions of the language.

Comment. $(\phi \rightarrow \psi)$ is not an expression of sentential logic, but it may be used to represent an expression of sentential logic.

well-formed formula *Definition.* A **WELL-FORMED FORMULA (WFF)** of sentential logic is any expression that accords with the following seven rules:

(1) A sentence letter standing alone is a wff.

atomic sentence [*Definition.* The sentence letters are the **ATOMIC SENTENCES** of the language of sentential logic.]

(2) If ϕ is a wff, then the expression denoted by $\sim\phi$ is also a wff.

negation [*Definition.* A wff of this form is known as a **NEGATION**, and $\sim\phi$ is known as the **NEGATION OF ϕ**.]

(3) If ϕ and ψ are both wffs, then the expression denoted by $(\phi \,\&\, \psi)$ is a wff.

conjunction [*Definition.* A wff of this form is known as a **CONJUNCTION**. ϕ and ψ are known as the left and right **CONJUNCTS**, respectively.]

(4) If ϕ and ψ are both wffs, then the expression denoted by $(\phi \vee \psi)$ is a wff.

disjunction [*Definition.* A wff of this form is known as a **DISJUNCTION.** ϕ and ψ are the left and right **DISJUNCTS**, respectively.]

(5) If ϕ and ψ are both wffs, then the expression denoted by $(\phi \rightarrow \psi)$ is a wff.

Comment. The reason for introducing the ideas of a sentence and a denial will be apparent when the rules of proof are introduced in section 1.4.

Paren. Rules

Exercise 1.2.1 Which of the following expressions are wffs? If an expression is a wff, say whether it is an atomic sentence, a conditional, a conjunction, a disjunction, a negation, or a biconditional. For the binary connectives, identify the component wffs (antecedent, consequent, conjuncts, disjuncts, etc.).

i*	A
ii*	$(A$
iii*	(A)
iv*	$(A \rightarrow B)$
v*	$(A \rightarrow (B)\ )$
vi*	$(A \rightarrow (B \rightarrow C))$
vii*	$((P \,\&\, Q) \rightarrow R)$
viii*	$((A \,\&\, B) \vee (C \rightarrow (D \leftrightarrow G)))$
ix*	$\sim(A \rightarrow B)$
x*	$\sim(P \rightarrow Q) \vee \sim(Q \,\&\, R)$
xi*	$\sim(A)$
xii*	$(\sim A) \rightarrow B$
xiii*	$(\sim(P \,\&\, P) \,\&\, (P \leftrightarrow (Q \vee \sim Q)))$
xiv*	$(\sim((B \vee P) \,\&\, C) \leftrightarrow ((D \vee \sim G) \rightarrow H))$
xv*	$(\sim(Q \vee \sim(B)) \vee (E \leftrightarrow (D \vee X)))$

Exception: binary-unary: I.E. $P \rightarrow \sim Q$ is ok