Objectives

By the end of this set of slides, you should be able to:

1. Recognize and understand continuous probability distribution functions
2. Recognize the uniform probability distribution and apply it appropriately
3. Recognize the exponential probability distribution and apply it appropriately
Random Variables

- **Random Variable**: A random variable describes the outcomes of a statistical experiment.

- Random variables can be of two types:
  - Discrete random variables: can take only a finite or countable number of outcomes. Example: number of patients with a particular disease.
  - Continuous random variables: take values within a specified interval or continuum. Example: height or weight.

- The values of a random variable can vary with each repetition of an experiment.

- We will use capitol letters to denote random variables and lower case letter to denote the value of a random variable.

- If $X$ is a random variable, then $X$ is written in words, and $x$ is given a number.
Continuous Random Variable Example

- Let $X$ = the number of minutes until the next earthquake in California
- Is $X$ a continuous random variable?
- The sample space $S = \{???\}$
- $X$ is in words and $x$ is a number
- The $x$ values are not countable (continuous or infinite) outcomes
- Because you cannot count the possible values that $X$ can take on and the outcomes are random, $X$ is a continuous random variable
- Other examples of continuous random variables?
Probability Distribution Function

- Just like a discrete random variable, every continuous random variable has an associated probability distribution function.
- The graph of a continuous probability distribution is a curve.
- Probability is represented by the area under the curve.
- Example: What is the probability that $X$ is between 8 and 11?
Probability Distribution Function

- In the discrete case, we could count all of the possible outcomes and assign each one a probability, in the continuous case WE CANNOT (why?)
- Instead of counting outcomes, we will be working now with area underneath the curves to solve probability problems
- The entire area under the curve is equal to one
- Probability will be found for intervals of $X$ rather than for individual $X$ values, i.e.,
  - $P(c < X < d)$ is the probability the random variable $X$ is in the interval between values $c$ and $d$
  - For EVERY continuous random variable, the probability that $P(X = c)$ is ALWAYS equal to 0, i.e., $P(X = c) = 0$ (why?)
- $P(c < X < d)$ is the same as $P(c \leq X \leq d)$ because probability is equal to area
We will find the area that represents the probability by using

- Geometry
- Formulas
- Technology
- Probability Tables

In general, calculus is needed to find the area under the curve for many probability distributions (in particular taking integrals).

We will NOT be using calculus in this class!

For continuous probability distribution functions,

\[
\text{PROBABILITY} = \text{AREA}
\]
Continuous Probability Functions

- Notation: We will use $f(x)$ to denote the probability density function (pdf)
- You may think of $f(x)$ as a function
- We define $f(x)$ so that the area between it and the x-axis is equal to a probability
- Since the maximum probability is one, the maximum area is also one
- For continuous probability distribution functions,

\[
\text{PROBABILITY} = \text{AREA}
\]
Example #1

- Consider the pdf \( f(x) = \frac{1}{20} \) for \( 0 \leq x \leq 20 \)
- The graph of \( f(x) \) is a horizontal line

• **Question**: What is the probability that \( 5 \leq X \leq 10 \)?
Example #1 Continued

- **Question:** What is the probability that $5 \leq X \leq 10$?

$P(5 \leq X \leq 10) = (10 - 5) \left( \frac{1}{20} \right) = 0.25$

- **Recall:** Area of a rectangle = (base)(height)
The Uniform Distribution

- The uniform distribution is a continuous probability distribution
- The uniform distribution describes data where the events are equally likely to occur
- Example: Age could be a continuous random variable that follows a uniform distribution
The Uniform Distribution

• Let $X$ be a uniform random variable
• $X \sim U(a, b)$
• The uniform probability density function is

$$f(x) = \frac{1}{b - a}$$

▷ $a =$ the lowest value of $x$
▷ $b =$ the highest value of $x$

• The probability that $c \leq X \leq d$ is

$$P(c \leq X \leq d) = \frac{d - c}{b - a}$$
Example #2

- Let $X \sim U(0, 23)$
- **Question:** Find the probability that $2 < X < 18$

**Solution:**

$$P(2 < X < 18) = \frac{18 - 2}{23 - 0} = \frac{16}{23}$$
Example #2 Continued

- Let $X \sim U(0, 23)$
- **Question:** Find the probability that $X > 12$ given that $X > 8$

Solution:

$$P(X > 12 | X > 8) = \frac{23 - 12}{23 - 8} = \frac{11}{15}$$
Example #2 Continued

- Let $X \sim U(0, 23)$
- **Question:** Find the probability that $X > 12$ given that $X > 8$
- Could we have solved it a different way?
- **Solution:**

$$P(X > 12 | X > 8) = \frac{P(X > 12 \text{ and } X < 18)}{P(X > 8)}$$

$$= \frac{P(X > 12)}{P(X > 8)}$$

$$= \frac{11}{23} \div \frac{15}{23}$$

$$= \frac{11}{15}$$
Expected Value (Mean) and Standard Deviation

- Let $X \sim U(a, b)$ then:
  - The expected value of $X$ is
    \[
    \mu = \frac{a + b}{2}
    \]
  - The standard deviation of $X$ is
    \[
    \sigma = \sqrt{\frac{(b - a)^2}{12}}
    \]
- Example: Let $X \sim U(0, 23)$, then
  \[
  \mu = \frac{23 + 0}{2} = 11.5
  \]
  and
  \[
  \sigma = \sqrt{\frac{(23 - 0)^2}{12}} = 6.639528
  \]
Percentiles for Uniform Random Variables

- **Recall**: The $P^{th}$ percentile divides the data between the lower $P\%$ and the upper $(100 - P)\%$ of the data:
- $P\%$ of data values are less than (or equal to) the $P^{th}$ percentile
- **Question**: How can we find the value $k$ corresponding to the $P^{th}$ percentile?
Percentiles for Uniform Random Variables

- **Question:** How can we find the value $k$ corresponding to the $P^{th}$ percentile?

- We know that $P(X \leq k) = P^{th}$ percentile, so

$$
\frac{k - a}{b - a} = P^{th} \text{ percentile}
$$

Rearranging terms we get

$$
k = a + (b - a)(P^{th} \text{ Percentile})
$$

- **Example:** Let $X \sim U(0, 23)$, what value corresponds to the $78^{th}$ percentile?

$$
k = 0 + (23 - 0)(0.78) = 17.94
$$

So 17.94 is the $78^{th}$ percentile
The Exponential Distribution

- The exponential distribution is a continuous probability distribution.
- The exponential distribution is often concerned with the amount of time until some specific event occurs.
- Some examples:
  - The amount of time until an earthquake occurs
  - The length (in minutes) of long distance business calls
  - The amount of time (in months) a car battery lasts

Plot of the Exponential Distribution

Life of Car Battery (in months)

Probability Density Function
The Exponential Distribution

- Let $X$ be a exponential random variable
- $X \sim \text{Exp}(m)$
- The exponential probability density function is

$$f(x) = me^{-mx}$$

▷ $m$ = the "decay rate"
▷ $m$ must be positive, i.e., $m > 0$
▷ $e = 2.71828182846...$
The Exponential Distribution

- Let $X$ be an exponential random variable
- $X \sim \text{Exp}(m)$
- The exponential probability density function is
  \[ f(x) = me^{-mx} \]
- The probability that $X \leq d$ is
  \[ P(X \leq d) = 1 - e^{-md} \]
- The probability that $c \leq X \leq d$ is
  \[ P(c \leq X \leq d) = e^{-mc} - e^{-md} \]
- The probability that $X \leq d$ is
  \[ P(X \geq d) = e^{-md} \]
Example #3

- Let $X \sim \text{Exp}(0.5)$
- **Question:** Find the probability that $2 < X < 4$

**Solution:**

$$P(2 < X < 4) = e^{-0.5(2)} - e^{-0.5(4)} = e^{-1} - e^{-2} = 0.2325442$$
Example #3 Continued

- Let $X \sim \text{Exp}(0.5)$
- **Question:** Find the probability that $X < 4$
- **Solution:**

$$P(X < 4) = 1 - e^{-0.5(4)} = 1 - e^{-2} = 0.8646647$$

- **Question:** Find the probability that $X > 4$
- **Solution:**

$$P(X > 4) = 1 - P(X < 4) = 1 - (1 - e^{-0.5(4)}) = e^{-2} = 0.1353353$$

- **Question:** Find the probability that $X = 4$
- **Solution:**

$$P(X = 4) = ??$$
Expected Value (Mean) and Standard Deviation

- Let $X \sim \text{Exp}(m)$, where $m > 0$, then:
  - The expected value of $X$ is
    \[ \mu = \frac{1}{m} \]
  - The standard deviation of $X$ is
    \[ \sigma = \frac{1}{m} \]

- Example: Let $X \sim \text{Exp}(4)$, then
  \[ \mu = \frac{1}{4} \]
  and
  \[ \sigma = \frac{1}{4} \]
Percentiles for Exponential Random Variables

- **Question:** How can we find the value $k$ corresponding to the $P^{th}$ percentile?
Percentiles for Exponential Random Variables

- **Question:** How can we find the value $k$ corresponding to the $P^{th}$ percentile?
- We know that $P(X \leq k) = P^{th}$ percentile, so

$$1 - e^{-km} = P^{th} \text{ percentile}$$

Rearranging terms we get

$$k = -\frac{\ln(1 - P^{th} \text{ Percentile})}{m}$$

- **Example:** Let $X \sim \text{Exp}(0.5)$, what value corresponds to the $90^{th}$ percentile?

$$k = -\frac{\ln(1 - 0.90)}{0.5} = -\frac{\ln(0.10)}{0.5} = 4.60517$$

So 4.60517 is the $90^{th}$ percentile
Other Continuous Probability Distributions

- Some other well known discrete probability distributions:
  - The normal (Gaussian) distribution
  - The student-t distribution
  - The $\chi^2$ distribution (chi-squared)
  - The gamma distribution
  - The beta distribution
  - The weibull distribution

- And the list goes on...