In 1609, after combing through massive amounts of astronomical data, Johannes Kepler published three laws of planetary motion.

I. A planet revolves around the sun in an elliptical orbit with the sun at one focus.

II. The line joining the sun to a planet sweeps out equal areas in equal times.

III. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

In this lab we will derive these same three laws without looking at any astronomical data. Instead we will use two of Newton's laws.

$$\text{Newton's Second Law of Motion:} \qquad \boldsymbol{F} = m\boldsymbol{a}$$

$$\text{Newton's Law of Gravitation:} \qquad \boldsymbol{F} = -\frac{GMm}{\|\boldsymbol{r}\|^3}\boldsymbol{r}$$

Where $\boldsymbol{r}(t)$ governs the motion of the planet, M and m are the masses of the sun and the planet respectively, and G is the gravitational constant.

1) First we will show that the path of the planet is a planar curve.

a) Use Newton's Laws to eliminate force and solve for acceleration. What does this tell you about the geometric relationship between $\boldsymbol{r}$ and $\boldsymbol{a}$? *(Please address the relationship*
I *in terms of both the magnitude and the direction of the vectors.)*

b) Prove that $\boldsymbol{r}\times\boldsymbol{v}$ is a constant vector. We will denote this vector by $\boldsymbol{h}$.

Since $\boldsymbol{h}$ is a constant vector orthogonal to both $\boldsymbol{r}$ and $\boldsymbol{r}'$ at all times, $\boldsymbol{r}$ lies on a plane. We may choose coordinates so that this motion takes place in the xy-plane with the sun at the origin and with $\boldsymbol{h}$ pointing in the positive z direction.

2) We will now examine several cross products in an attempt to find a suitable choice for the direction of the x-axis. To aid us in this endeavor we will decompose $\boldsymbol{r}$ into its direction and magnitude. We will use the notation:

$$\boldsymbol{u} = \frac{\boldsymbol{r}}{\|\boldsymbol{r}\|} \qquad \text{and} \qquad r = \|\boldsymbol{r}\|$$

a) Prove $\boldsymbol{h} = r^2(\boldsymbol{u}\times\boldsymbol{u}')$.

b) Use questions 1(a) and 2(a) to prove $\boldsymbol{a}\times\boldsymbol{h} = GM\boldsymbol{u}'$. *(Hint: You will need to Property 11.6 in Section 12.4 of your textbook and the fact that $\boldsymbol{u}$ is a unit vector.)*

c) Prove $\boldsymbol{v}\times\boldsymbol{h} = GM\boldsymbol{u} + \boldsymbol{c}$, where $\boldsymbol{c}$ is a constant vector. *(Hint: Differentiate the cross product and apply the results of 2(b).)*

By definition we see that both $\boldsymbol{v}\times\boldsymbol{h}$ and $\boldsymbol{u}$ are orthogonal to $\boldsymbol{h}$**.** Therefore the result of 2(c) tells us the vector $\boldsymbol{c}$ is orthogonal to $\boldsymbol{h}$. Hence it lies in our xy-plane. We will choose our x-axis to lie in the direction of $\boldsymbol{c}$.

3) Let θ be the angle between $\boldsymbol{c}$ and $\boldsymbol{r}$. Then (r, θ) are the polar coordinates of the planet.

> a) Use the triple product to show $\boldsymbol{r} \cdot (\boldsymbol{v}\times\boldsymbol{h}) = h^2$, where $h = \|\boldsymbol{h}\|$.
>
> b) Use 2(c) to show $\boldsymbol{r} \cdot (\boldsymbol{v}\times\boldsymbol{h}) = GMr + rc\cos\theta$, where $c = \|\boldsymbol{c}\|$.
>
> c) Equate 3(a) and 3(b), then solve for r. You should recognize your result as a conic section from Lab 4. What is the eccentricity and directerix for this conic section? Without discussing the eccentricity, explain why this must be an ellipse and not a hyperbola or parabola.

Now that <u>we have proved Kepler's First Law of Planetary Motion</u> we will continue on to the other two.

4) Using the results above we know that $\boldsymbol{r} = \langle r\cos\theta, r\sin\theta\,, 0\rangle$ where both r and θ are functions of time t.

> a) Find $\boldsymbol{h}$.
>
> b) We know from polar coordinates that the area function for the path of the planet is
>
> $$A = \int_0^\theta \frac{1}{2} r^2 d\theta$$
>
> Use the Fundamental Theorem of Calculus and the chain rule to find $\frac{dA}{dt}$.
>
> c) Combine 4(a) and 4(b) to show that $\frac{dA}{dt} = \frac{1}{2}h$ and is therefore constant. <u>This proves Kepler's Second Law of Planetary Motion</u>.

5) Let T be the time it takes the planet to make one orbit around the sun (i.e. T is the period of revolution of the planet). Suppose that the lengths of the major and minor axes of the ellipse are $2a$ and $2b$.

> a) Find the area of the ellipse in two ways.
>
> - First integrate 4(c).
> - Then find the area enclosed by the curve $\boldsymbol{r}(t) = \langle a\cos t\,, b\sin t\rangle$.
>
> Equating these, show that $T = \frac{2\pi ab}{h}$.

b) Show that $\frac{h^2}{GM} = ed = \frac{b^2}{a}$. *(Hint: The first equality should come from 3(c). For the second equality, use the final result of Lab 4.)*

c) Use 5(a) and 5(b) to show that $T^2 = \frac{4\pi^2}{GM} a^3$. Note that not only does this prove Kepler's Third Law of Planetary Motion, but it also show that the constant of proportionality $\frac{4\pi^2}{GM}$ is independent of the mass of the planet.