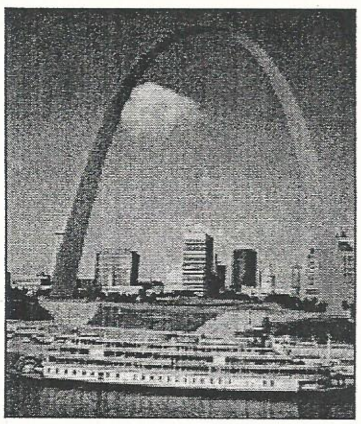
are essentially the same. For example, the intervals $(0, 1)$, $(0, \pi)$ and $(-\pi/2, \pi/2)$ are all finite open intervals. They are *homeomorphic*. By definition, intervals I_1 and I_2 are homeomorphic if there is a function f from I_1 onto I_2 that has an inverse with f and f^{-1} both being continuous. A homeomorphism from $(0, 1)$ to $(0, \pi)$ is $f(x) = \pi x$. Show that this is a homeomorphism by finding its inverse and verifying that both are continuous. Find a homeomorphism from $(0, \pi)$ to $(-\pi/2, \pi/2)$. (Hint: Sketch a picture, and decide how you could move the interval $(0, \pi)$ to produce the interval $(-\pi/2, \pi/2)$.) This will take some thinking, but try to find a homeomorphism for any two finite open intervals (a, b) and (c, d). It remains to decide whether the interval $(-\infty, \infty)$ is different because it is infinite or the same because it is open. In fact, $(-\infty, \infty)$ is homeomorphic to $(-\pi/2, \pi/2)$ and hence to all other open intervals. Show that $\tan^{-1} x$ is a homeomorphism from $(-\infty, \infty)$ to $(-\pi/2, \pi/2)$.

6.9 THE HYPERBOLIC FUNCTIONS



The Gateway Arch, St. Louis, MO

The Gateway Arch in Saint Louis is one of the most distinctive and recognizable architectural structures in the United States. There are several surprising features of its shape. For instance, is the arch taller than it is wide? Most people think that it is taller, but this is the result of a common optical illusion. In fact, the arch has the same width as height. A slightly less mysterious illusion of the arch's shape is that it is **not** a parabola. Its shape corresponds to the graph of the hyperbolic cosine function (called a **catenary**). This function and the other five hyperbolic functions are introduced in this section.

You may be wondering why we need more functions. Well, these functions are not entirely new. They are simply common combinations of exponentials. We study them because of their usefulness in applications (e.g., the Gateway Arch) and their convenience in solving equations (in particular, differential equations).

The **hyperbolic sine** function is defined by

$$\boxed{\sinh x = \frac{e^x - e^{-x}}{2},}$$

for all $x \in (-\infty, \infty)$. The **hyperbolic cosine** function is defined by

$$\boxed{\cosh x = \frac{e^x + e^{-x}}{2},}$$

again for all $x \in (-\infty, \infty)$. You can easily use the preceding definitions to verify the important identity

$$\cosh^2 u - \sinh^2 u = 1, \tag{9.1}$$

for any value of u. (We leave this as an exercise.) In light of this identity, notice that if $x = \cosh u$ and $y = \sinh u$, then

$$x^2 - y^2 = \cosh^2 u - \sinh^2 u = 1,$$

which you should recognize as the equation of a hyperbola. This identity is the source of the name "hyperbolic" for these functions. You should also notice some parallel with the trigonometric functions $\cos x$ and $\sin x$. This will become even more apparent with what follows.

The remaining four hyperbolic functions are defined in terms of the hyperbolic sine and hyperbolic cosine functions, in a manner analogous to their trigonometric counterparts. That is, we define the **hyperbolic tangent** function $\tanh x$, the **hyperbolic cotangent** function $\coth x$, the **hyperbolic secant** function $\operatorname{sech} x$ and the **hyperbolic cosecant** function $\operatorname{csch} x$ as follows:

$$\boxed{\begin{aligned} \tanh x &= \frac{\sinh x}{\cosh x}, & \coth x &= \frac{\cosh x}{\sinh x} \\ \operatorname{sech} x &= \frac{1}{\cosh x}, & \operatorname{csch} x &= \frac{1}{\sinh x}. \end{aligned}}$$

These functions are remarkably easy to deal with, and we can readily determine their behavior, using what we have already learned about exponentials. First, note that

$$\boxed{\frac{d}{dx}\sinh x = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x.}$$

Similarly, we can establish the remaining derivative formulas:

$$\boxed{\begin{aligned} &\frac{d}{dx}\cosh x = \sinh x, & &\frac{d}{dx}\tanh x = \operatorname{sech}^2 x \\ &\frac{d}{dx}\coth x = -\operatorname{csch}^2 x, & &\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x \\ \text{and } &\frac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x. \end{aligned}}$$

These are all elementary applications of earlier derivative rules and are left as exercises. As it turns out, only the first three of these are of much significance.

Example 9.1 Computing the Derivative of a Hyperbolic Function

Compute the derivative of $f(x) = \sinh^2(3x)$.

Solution From the chain rule, we have

$$\begin{aligned} f'(x) = \frac{d}{dx}\sinh^2(3x) &= \frac{d}{dx}[\sinh(3x)]^2 \\ &= 2\sinh(3x)\frac{d}{dx}[\sinh(3x)] \\ &= 2\sinh(3x)\cosh(3x)\frac{d}{dx}(3x) \\ &= 2\sinh(3x)\cosh(3x)(3) \\ &= 6\sinh(3x)\cosh(3x). \end{aligned}$$

Example 9.2 An Integral Involving a Hyperbolic Function

Evaluate $\int x \cosh(x^2)\, dx$.

Solution Notice that you can evaluate this integral using a substitution. If we let $u = x^2$, we get $du = 2x\, dx$ and so,

$$\int x \cosh(x^2)\, dx = \frac{1}{2}\int \underbrace{\cosh(x^2)}_{\cosh u}\,\underbrace{(2x)\, dx}_{du}$$
$$= \frac{1}{2}\int \cosh u\, du = \frac{1}{2}\sinh u + c$$
$$= \frac{1}{2}\sinh(x^2) + c.$$

For $f(x) = \sinh x$, note that

$$f(x) = \sinh x = \frac{e^x - e^{-x}}{2} \begin{cases} > 0 & \text{if } x > 0 \\ < 0 & \text{if } x < 0 \end{cases}.$$

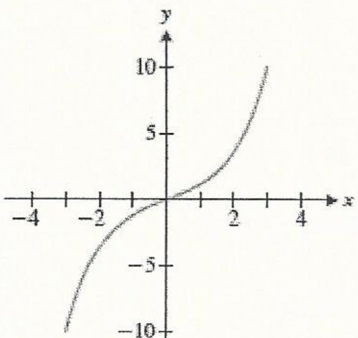


Figure 6.45
$y = \sinh x$.

This is left as an exercise. Further, since $f'(x) = \cosh x > 0$, $\sinh x$ is increasing for all x. Next, note that $f''(x) = \sinh x$. Thus, the graph is concave down for $x < 0$ and concave up for $x > 0$. Finally, you can easily verify that

$$\lim_{x\to\infty} \sinh x = \infty \quad \text{and} \quad \lim_{x\to-\infty} \sinh x = -\infty.$$

It is now a simple matter to produce the graph seen in Figure 6.45. Similarly, you should be able to produce the graphs of $\cosh x$ and $\tanh x$ seen in Figures 6.46a and 6.46b, respectively. We leave the graphs of the remaining three hyperbolic functions to the exercises.

If a flexible cable or wire (such as a power line or telephone line) hangs between two towers, it will assume the shape of a catenary curve (derived from the Latin word *catena* meaning "chain"). As we will show at the end of this section, this naturally occurring curve corresponds to the graph of the hyperbolic cosine function $f(x) = a\cosh\left(\frac{x}{a}\right)$.

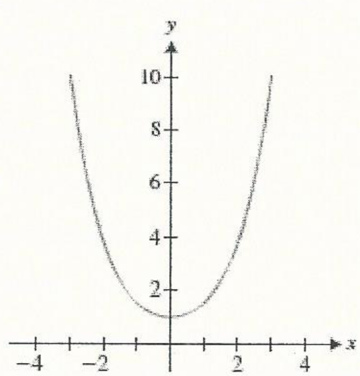


Figure 6.46a
$y = \cosh x$.

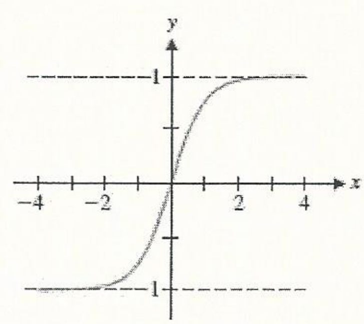


Figure 6.46b
$y = \tanh x$.

Example 9.3 Finding the Amount of Sag in a Hanging Cable

For the catenary $f(x) = 20\cosh(\frac{x}{20})$, for $-20 \le x \le 20$, find the amount of sag in the cable and the arc length.

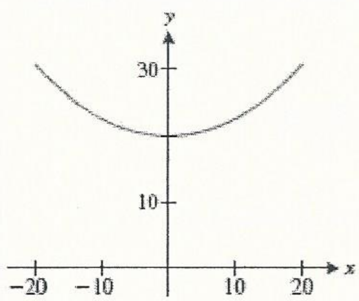


Figure 6.47
$y = 20\cosh(\frac{x}{20})$.

Solution From the graph of the function in Figure 6.47, it appears that the minimum value of the function is at the midpoint $x = 0$, with the maximum at $x = -20$ and $x = 20$. To verify this observation, note that

$$f'(x) = \sinh\left(\frac{x}{20}\right)$$

and hence, $f'(0) = 0$, while $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$, for $x > 0$. Thus, f decreases to a minimum at $x = 0$. Further, $f(-20) = f(20) \approx 30.86$ is the maximum for $-20 \le x \le 20$ and $f(0) = 20$, so that the cable sags approximately 10.86 feet. From the usual formula for arc length, developed in section 5.4, the length of the cable is

$$L = \int_{-20}^{20} \sqrt{1 + [f'(x)]^2}\,dx = \int_{-20}^{20} \sqrt{1 + \sinh^2\left(\frac{x}{20}\right)}\,dx.$$

Notice that from (9.1), we have

$$1 + \sinh^2 x = \cosh^2 x.$$

Using this identity, the arc length integral simplifies to

$$\begin{aligned} L &= \int_{-20}^{20} \sqrt{1 + \sinh^2\left(\frac{x}{20}\right)}\,dx = \int_{-20}^{20} \cosh\left(\frac{x}{20}\right)dx \\ &= 20\sinh\left(\frac{x}{20}\right)\Big|_{-20}^{20} = 20[\sinh(1) - \sinh(-1)] \\ &= 40\sinh(1) \approx 47 \text{ feet.} \end{aligned}$$

The Inverse Hyperbolic Functions

You should note from the graphs of $\sinh x$ and $\tanh x$ that these functions are one-to-one. Also, $\cosh x$ is one-to-one for $x \ge 0$. Thus, we can define inverses for these functions, as follows. For any $x \in (-\infty, \infty)$, we define the **inverse hyperbolic sine** by

$$y = \sinh^{-1} x \quad \text{if and only if } \sinh y = x.$$

For any $x \ge 1$, we define the **inverse hyperbolic cosine** by

$$y = \cosh^{-1} x \quad \text{if and only if } \cosh y = x, \text{ and } y \ge 0.$$

Finally, for any $x \in (-1, 1)$, we define the **inverse hyperbolic tangent** by

$$y = \tanh^{-1} x \quad \text{if and only if } \tanh y = x.$$

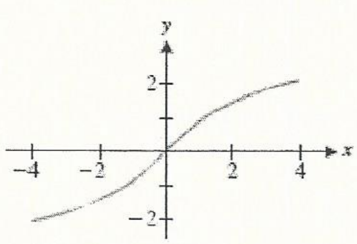


Figure 6.48a
$y = \sinh^{-1} x$.

Inverses for the remaining three hyperbolic functions can be defined similarly and are left to the exercises. We show the graphs of $y = \sinh^{-1} x$, $y = \cosh^{-1} x$ and $y = \tanh^{-1} x$ in Figures 6.48a, 6.48b and 6.48c, respectively. (As usual, you can obtain these by reflecting the graph of the original function through the line $y = x$.)

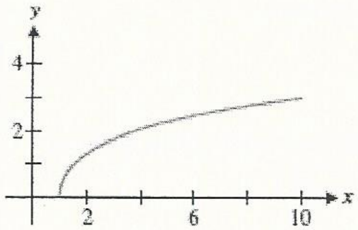


Figure 6.48b
$y = \cosh^{-1} x$.

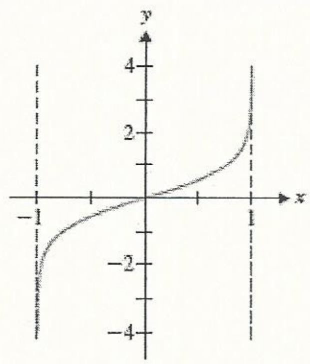


Figure 6.48c
$y = \tanh^{-1} x$.

We can find derivatives for the inverse hyperbolic functions using implicit differentiation, just as we have for previous inverse functions. We have that

$$y = \sinh^{-1} x \quad \text{if and only if } \sinh y = x. \tag{9.2}$$

Differentiating both sides of this last equation with respect to x yields

$$\frac{d}{dx} \sinh y = \frac{d}{dx} x$$

or

$$\cosh y \frac{dy}{dx} = 1.$$

Solving for the derivative, we find

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}},$$

since we know that

$$\cosh^2 y - \sinh^2 y = 1,$$

from (9.1). That is, we have shown that

$$\boxed{\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{1 + x^2}}.}$$

Note the similarity with the derivative formula for $\sin^{-1} x$. We can likewise establish derivative formulas for the other five inverse hyperbolic functions. We list these below for the sake of completeness.

$$\boxed{\begin{array}{ll} \dfrac{d}{dx} \sinh^{-1} x = \dfrac{1}{\sqrt{1 + x^2}} & \dfrac{d}{dx} \cosh^{-1} x = \dfrac{1}{\sqrt{x^2 - 1}} \\ \dfrac{d}{dx} \tanh^{-1} x = \dfrac{1}{1 - x^2} & \dfrac{d}{dx} \coth^{-1} x = \dfrac{1}{1 - x^2} \\ \dfrac{d}{dx} \operatorname{sech}^{-1} x = \dfrac{-1}{x\sqrt{1 - x^2}} & \dfrac{d}{dx} \operatorname{csch}^{-1} x = \dfrac{-1}{|x|\sqrt{1 + x^2}} \end{array}}$$

Before closing this section, we wish to point out that the inverse hyperbolic functions have a significant advantage over earlier inverse functions we have discussed. It turns out that we can solve for the inverse functions explicitly in terms of more elementary functions.

Example 9.4 Finding a Formula for an Inverse Hyperbolic Function

Find an explicit formula for $\sinh^{-1} x$.

Solution Recall from (9.2) that

$$y = \sinh^{-1} x \quad \text{if and only if } \sinh y = x.$$

Using this definition, we have

$$x = \sinh y = \frac{e^y - e^{-y}}{2}. \tag{9.3}$$

We can solve this equation for y, as follows. First, recall also that

$$\cosh y = \frac{e^y + e^{-y}}{2}.$$

Now, notice that adding these last two equations and using the identity (9.1), we have

$$\begin{aligned} e^y &= \sinh y + \cosh y = \sinh y + \sqrt{\cosh^2 y} \\ &= \sinh y + \sqrt{\sinh^2 y + 1} \\ &= x + \sqrt{x^2 + 1}, \end{aligned}$$

from (9.3). Finally, taking the natural logarithm of both sides, we get

$$y = \ln(e^y) = \ln\left(x + \sqrt{x^2+1}\right).$$

That is, we have found a formula for the inverse hyperbolic sine function:

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right).$$

Similarly, we can show that for $x \geq 1$,

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$$

and for $-1 < x < 1$,

$$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right).$$

We leave it to the exercises to derive these formulas and corresponding formulas for the remaining inverse hyperbolic functions. There is little point in memorizing any of these formulas. You need only realize that these are always available by performing some elementary algebra.

Derivation of the Catenary

We close this section by deriving a formula for the catenary. As you follow the steps, pay special attention to the variety of calculus results that we use.

In Figure 6.49, we assume that the lowest point of the catenary curve is located at the origin. We further assume that the cable has constant linear density ρ (measured in units of weight per unit length) and that the function $y = f(x)$ is twice continuously differentiable. We focus on the portion of the cable from the origin to the general point (x, y) indicated in the figure. Since this section is not moving, the horizontal and vertical forces must be balanced. Horizontally, this section of cable is pulled to the left by the tension H at the origin and is pulled to the right by the horizontal component $T\cos\theta$ of the tension T at the point (x, y). Notice that these forces are balanced if

$$H = T\cos\theta. \tag{9.4}$$

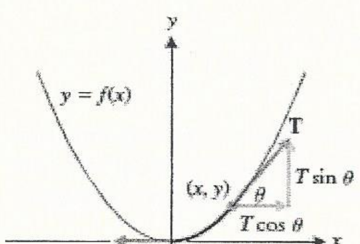


Figure 6.49
Forces acting on a section of hanging cable.

Vertically, the section of cable is pulled up by the vertical component $T\sin\theta$ of the tension. The section of cable is pulled down by the weight of the section. Notice that the weight of the section is given by the product of the density ρ (weight per unit length) and the length of the section. Recall from your study of arc length in Chapter 5 that the arc length of this section of cable is given by $\int_0^x \sqrt{1+[f'(t)]^2}\,dt$. So, the vertical forces will balance if

$$T\sin\theta = \rho \int_0^x \sqrt{1+[f'(t)]^2}\,dt. \tag{9.5}$$

We can combine equations (9.4) and (9.5) by multiplying (9.4) by $\tan\theta$ to get $H\tan\theta = T\sin\theta$, then using (9.5) to conclude that

$$H\tan\theta = \rho\int_0^x \sqrt{1+[f'(t)]^2}\,dt.$$

Notice from Figure 6.49 that $\tan\theta = f'(x)$, so that we have

$$Hf'(x) = \rho\int_0^x \sqrt{1+[f'(t)]^2}\,dt.$$

Differentiating both sides of this equation, the Fundamental Theorem of Calculus gives us

$$Hf''(x) = \rho\sqrt{1+[f'(x)]^2}. \qquad (9.6)$$

Now, divide both sides of the equation by H and name $b = \frac{\rho}{H}$. Further, substitute $u(x) = f'(x)$. Equation (9.6) then becomes

$$u'(x) = b\sqrt{1+[u(x)]^2},$$

which you should recognize as a separable differential equation. Putting together all of the u terms and integrating with respect to x gives us

$$\int \frac{1}{\sqrt{1+[u(x)]^2}}u'(x)\,dx = \int b\,dx.$$

You should recognize the integral on the left-hand side as $\sinh^{-1}(u(x))$, so that we now have

$$\sinh^{-1}(u(x)) = bx + c.$$

Notice that since $f(x)$ has a minimum at $x = 0$, we must have that $u(0) = f'(0) = 0$. So, taking $x = 0$, we get $c = \sinh^{-1}(0) = 0$. From $\sinh^{-1}(u(x)) = bx$, we obtain $u(x) = \sinh(bx)$. Now, recall that $u(x) = f'(x)$, so that $f'(x) = \sinh(bx)$. Integrating this gives us

$$\begin{aligned} f(x) &= \int \sinh(bx)\,dx \\ &= \tfrac{1}{b}\cosh(bx) + c. \end{aligned}$$

Finally, recall that $f(0) = 0$ and so, we must have $c = \frac{-1}{b}$. This leaves us with $f(x) = \frac{1}{b}\cosh(bx)\frac{-1}{b}$. Finally, writing $a = \frac{1}{b}$, we obtain the catenary equation $f(x) = a\cosh\left(\frac{x}{a}\right) - a$.

EXERCISES 6.9

1. Compare the derivatives and integrals of the trigonometric functions to the derivatives and integrals of the hyperbolic functions. Also note that the trigonometric identity $\cos^2 x + \sin^2 x = 1$ differs only by a minus sign from the corresponding hyperbolic identity $\cosh^2 x - \sinh^2 x = 1$.

2. As noted in the text, the hyperbolic functions are not really new functions. They provide names for useful combinations of exponential functions. Explain why it is advantageous to assign special names to these functions instead of leaving them as exponentials.

3. Briefly describe the graphs of $\sinh x$, $\cosh x$ and $\tanh x$. Which simple polynomials do the graphs of $\sinh x$ and $\cosh x$ resemble?

4. The catenary (hyperbolic cosine) is the shape assumed by a hanging cable because this distributes the weight of the

cable most evenly throughout the cable. Knowing this, why was it smart to build the Gateway Arch in this shape? Why would you suspect that the profile of an egg has this same shape?

In exercises 5–12, sketch the graph of each function.

5. $\cosh 2x$
6. $\sinh 3x$
7. $\tanh 4x$
8. $\tanh x^2$
9. $\cosh 2x \sinh 2x$
10. $e^{-x} \sinh x$
11. $x^2 \sinh 2x$
12. $x^3 \sinh x$

In exercises 13–24, find the derivative of each function.

13. $\cosh 4x$
14. $\cosh x^2$
15. $\sinh 2x$
16. $\sinh \sqrt{x}$
17. $\tanh 4x$
18. $\tanh x^2$
19. $\cosh^{-1} 2x$
20. $\sinh^{-1} 3x$
21. $x^2 \sinh 2x$
22. $x^3 \sinh x$
23. $\tanh^{-1} 4x$
24. $\sinh^{-1} x^2$

In exercises 25–36, evaluate each integral.

25. $\int \cosh 6x \, dx$
26. $\int \sinh 2x \, dx$
27. $\int \tanh 3x \, dx$
28. $\int \operatorname{sech}^2 x \, dx$
29. $\int_0^1 \frac{e^{4x} - e^{-4x}}{2} \, dx$
30. $\int_0^1 \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} \, dx$
31. $\int \frac{2}{\sqrt{1+x^2}} \, dx$
32. $\int \frac{2x}{\sqrt{1+x^4}} \, dx$
33. $\int \cos x \sinh(\sin x) \, dx$
34. $\int x \cosh(x^2) \, dx$
35. $\int_0^1 \cosh x \, e^{\sinh x} \, dx$
36. $\int_0^1 \frac{\cosh 2x}{3 + \sinh 2x} \, dx$

37. Derive the formulas $\frac{d}{dx} \cosh x = \sinh x$ and $\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$.

38. Derive the formulas for the derivatives of $\coth x$, $\operatorname{sech} x$ and $\operatorname{csch} x$.

39. Using the properties of exponential functions, prove that $\sinh x > 0$ if $x > 0$ and $\sinh x < 0$ if $x < 0$.

40. Use the first and second derivatives to explain the properties of the graph of $\tanh x$.

41. Use the first and second derivatives to explain the properties of the graph of $\cosh x$.

42. Prove that $\cosh^2 x - \sinh^2 x = 1$.

43. Find an explicit formula, as in example 9.4, for $\cosh^{-1} x$.

44. Find an explicit formula, as in example 9.4, for $\tanh^{-1} x$.

45. Suppose that a hanging cable has the shape $10 \cosh(x/10)$ for $-20 \le x \le 20$. Find the amount of sag in the cable.

46. Find the length of the cable in exercise 45.

47. Suppose that a hanging cable has the shape $15 \cosh(x/15)$ for $-25 \le x \le 25$. Find the amount of sag in the cable.

48. Find the length of the cable in exercise 47.

49. Suppose that a hanging cable has the shape $a \cosh(x/a)$ for $-b \le x \le b$. Show that the amount of sag is given by $a \cosh(b/a) - a$ and the length of the cable is $2a \sinh(b/a)$.

50. Show that $\cosh(-x) = \cosh x$ (i.e., $\cosh x$ is an even function) and $\sinh(-x) = -\sinh x$ (i.e., $\sinh x$ is an odd function).

51. Show that $e^x = \cosh x + \sinh x$. In fact, we will show that this is the only way to write e^x as the sum of even and odd functions. To see this, assume that $e^x = f(x) + g(x)$, where f is even and g is odd. Show that $e^{-x} = f(x) - g(x)$. Adding equations and dividing by two, conclude that $f(x) = \cosh x$. Then conclude that $g(x) = \sinh x$.

52. Show that both $\cosh x$ and $\sinh x$ are solutions of the differential equation $y'' - y = 0$. By comparison, show that both $\cos x$ and $\sin x$ are solutions of the differential equation $y'' + y = 0$.

53. Show that $\lim_{x\to\infty} \tanh x = 1$ and $\lim_{x\to-\infty} \tanh x = -1$.

54. Show that $\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}$.

55. In this exercise, we solve the initial value problem for the vertical velocity $v(t)$ of a falling object subject to gravity and air drag. Assume that $mv'(t) = -mg + kv^2$ for some positive constant k.

(a) Rewrite the equation as $\frac{1}{v^2 - mg/k} v'(t) = \frac{k}{m}$.

(b) Use the identity $\frac{1}{v^2 - a^2} = \frac{1}{2a}\left(\frac{1}{v-a} - \frac{1}{v+a}\right)$ with $a = \sqrt{\frac{mg}{k}}$ to solve the equation in (a).

(c) Show that $v(t) = -\sqrt{\frac{mg}{k}} \frac{ce^{2\sqrt{kg/m}t} - 1}{ce^{2\sqrt{kg/m}t} + 1}$.

(d) Use the initial condition $v(0) = 0$ to show that $c = 1$.

(e) Use the result of exercise 54 to conclude that

$$v(t) = -\sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{kg}{m}}\,t\right).$$

(f) Find the terminal velocity by computing $\lim_{t \to \infty} v(t)$.

56. Integrate the velocity function in exercise 55 part (e) to find the distance fallen in t seconds.

57. Two skydivers of weight 800 N drop from a height of 1000 m. The first skydiver dives head-first with a drag coefficient of $k = \frac{1}{8}$. The second skydiver is in a spread-eagle position with $k = \frac{1}{2}$. Compare the terminal velocities and the distances fallen in 2 seconds; 4 seconds.

58. A skydiver with an open parachute has terminal velocity 5 m/s. If the weight is 800 N, determine the value of k.

59. Long and Weiss derive the following equation for the horizontal velocity of the space shuttle during re-entry (see section 4.1): $v(t) = 7901 \tanh(-0.00124t + \tanh^{-1}(v_0/7901))$ m/s, where v_0 is the velocity at time $t = 0$. Find the maximum acceleration experienced by the shuttle from this horizontal motion (i.e., maximize $|v'(t)|$).

60. Graph the velocity function in exercise 59 with $v_0 = 2000$. Estimate the time t at which $v(t) = 0$.

61. The Saint Louis Gateway Arch is both 630 feet wide and 630 feet tall. Its shape looks very much like a parabola, but is actually a hyperbolic cosine. You will explore the difference between the two functions in this exercise. First, consider the model $y = 757.7 - 127.7\cosh(x/127.7)$ for $y \geq 0$. Find the x- and y-intercepts and show that this model (approximately) matches the arch's measurements of 630 feet wide and 630 feet tall. What would the 127.7 in the model have to be to match the measurements exactly? Now, consider a parabolic model. To have x-intercepts $x = -315$ and $x = 315$, explain why the model must have the form $y = -c(x + 315)(x - 315)$ for some positive constant c. Then find c to match the desired y-intercept of 630. Graph the parabola and hyperbolic cosine on the same axes for $-315 \leq x \leq 315$. How much difference is there between the graphs? Find the maximum distance between the curves. The authors have seen mathematics books where the arch is modeled by a parabola. How wrong is it to do this?

62. Suppose a person jumps out of an airplane from a great height. There are two primary forces acting on the skydiver: gravity and air resistance. In this situation, the air resistance would be proportional to the square of the velocity. Then an equation for the (downward) velocity would be $v' = g - cv^2$, where g is the gravitational constant and c is a constant determined by the orientation of the jumper's body. Replace c with g/v_T^2 and explain why the initial condition $v(0) = 0$ is reasonable. Then show that the solution of the IVP can be written in the form $v = v_T \tanh(gt/v_T)$. Show that v, the downward velocity, is an increasing function and find the limiting velocity, usually called the **terminal velocity**, as $t \to \infty$. As mentioned above, the constant c depends on the position of the jumper's body. If spread-eagle represents a c-value four times as large as a head-first dive, compare the corresponding terminal velocities. You may have seen video of skydivers jumping out of a plane at different times but catching up to each other and forming a circle. Explain how one diver could catch up to someone who jumped out of the plane earlier. Now, integrate the velocity function to obtain the distance function. Finally, answer the following two-part question. How much time and height does it take for a skydiver to reach 90% of terminal velocity?

CHAPTER REVIEW EXERCISES

In exercises 1–16, find the derivative of the function.

1. $\ln(x^3 + 5)$
2. $\ln(1 - \sin x)$
3. $\ln\sqrt{x^4 + x}$
4. $\ln\dfrac{x^2 - 1}{x^3 + 2x + 1}$
5. e^{-x^2}
6. $e^{\tan x}$
7. 4^{x^3}
8. 3^{1-2x}
9. $\sin^{-1} 2x$
10. $\cos^{-1} x^2$
11. $\tan^{-1}(\cos 2x)$
12. $\sec^{-1}(3x^2)$
13. $\cosh\sqrt{x}$
14. $\sinh(e^{2x})$
15. $\sinh^{-1} 3x$
16. $\tanh^{-1}(3x + 1)$

In exercises 17–40, evaluate the integral.

17. $\displaystyle\int \frac{x^2}{x^3 + 4}\,dx$
18. $\displaystyle\int \frac{e^{2x}}{e^{2x} + 4}\,dx$
19. $\displaystyle\int_0^1 \frac{x}{x^2 + 1}\,dx$
20. $\displaystyle\int_{\pi/12}^{\pi/4} \frac{\cos 2x}{\sin 2x}\,dx$