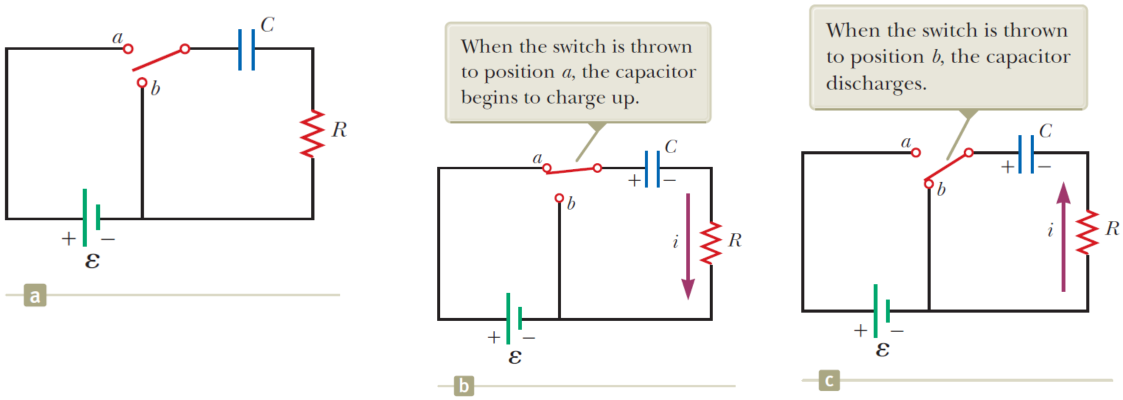
RC Circuits

So far we have considered DC circuits with constant current (magnitude and direction). In DC circuits containing capacitors, the current will remain in the same direction but the magnitude will vary with time. A circuit containing a capacitor and resistor connected in series is called an RC circuit.



In such an RC circuit the current is not constant and will vary with time.

CHARGING CAPACITOR

Initially the capacitor is uncharged with the switch open. There is no current since it's an open circuit. At t = 0 the switch is closed and the capacitor begins to charge by the battery (explain!). At some later time 't' the charge on the capacitor is 'q'. Applying the loop rule gives:

Applying loop rule:

$$(1)\quad \boxed{+V - \frac{q}{C} - iR = 0}$$

Let's use Eq. 1 to find the initial current and max charge on capacitor:

At t = 0, q = 0, and $I = I_o$

$V - 0/q - I_o R = 0$

$$\boxed{I_o = \frac{V}{R}}$$ current @ t=0 (max. current)

At t = 0 the potential across the battery appears entirely across the resistor. When the capacitor reaches its max. charge Q, I = 0. Using Eq. 1:

$$V - \frac{Q_{max}}{C} - 0 = 0$$

(2) $$\boxed{Q_{max} = CV}$$ (max. charge)

Let's now derive $q(t)$ and $i(t)$:

$$V - \frac{q}{C} - iR = 0$$

$$i = \frac{V}{R} - \frac{q}{RC} = \frac{CV}{RC} - \frac{q}{RC} = -\frac{1}{RC}(q - CV)$$

$$\frac{dq}{dt} = -\frac{1}{RC}(q - CV)$$

$$\frac{dq}{q - CV} = -\frac{1}{RC}\,dt$$

$$\int_0^q \frac{dq}{q - CV} = -\frac{1}{RC}\int_0^t dt$$

let $u = q - CV$

$du = dq$

$u_i = -CV$

$u_f = q - CV$

$$\int_{-CV}^{q-CV} \frac{du}{u} = -\frac{1}{RC}t$$

$$\ln u \Big|_{-CV}^{q-CV} = -\frac{1}{RC}t$$

$$\ln\left(\frac{q - CV}{-CV}\right) = -\frac{1}{RC}t$$

$$\ln\left(-\frac{q}{CV} + 1\right) = -\frac{1}{RC}t$$

$$-\frac{q}{CV} + 1 = e^{-t/RC}$$

$$\frac{q}{CV} = 1 - e^{-t/RC}$$

$$q = CV(1 - e^{-t/RC})$$

$$\boxed{q(t) = Q_{max}(1 - e^{-t/RC})}$$

Capacitor charge

$$i(t) = \frac{dq}{dt} = \frac{d}{dt}\left[Q_{max} - Q_{max}e^{-t/RC}\right]$$

$$i(t) = \left(-\frac{1}{RC}\right)(-Q)e^{-t/RC}$$

$$i(t) = \frac{\cancel{C}V}{R\cancel{C}}e^{-t/RC}$$

$$\boxed{i(t) = I_0 e^{-t/\tau}}$$

circuit current when charging capacitor

$$\boxed{\tau = RC}$$

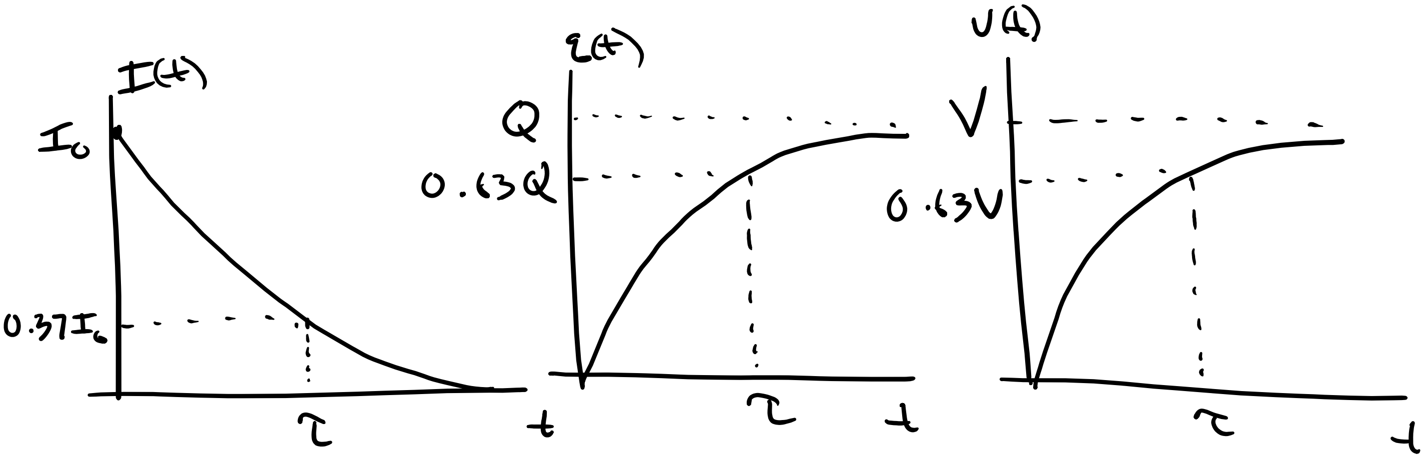
Time-constant

$$C = \frac{q(t)}{V(t)}$$

$$V(t) = \frac{q(t)}{C} = \frac{Q_{max}(1 - e^{-t/\tau})}{C}$$

$$\boxed{V(t) = V(1 - e^{-t/\tau})}$$

Voltage across capacitor while charging



$\tau = RC$ (Time-constant)

$q(\tau) = Q(1-e^{-1}) = \boxed{0.63Q}$

$V(\tau) = \boxed{0.63V}$

$I(\tau) = I_0 e^{-1} = \boxed{0.37 I_0}$

a) $\tau = RC$ is a measure of how fast a capacitor charges!

a) When $\tau = RC$ is small, the capacitor charges fast.

b) when $\tau = RC$ is large, the capacitor charges slow.

c) If R is small, then the easier the current flows, and thus small τ and capacitor charges faster.

Discharging Capacitor

After the full capacitor is fully charged and switch is placed on "b", then capacitor discharges thru resistor. If switch is placed to "b" at t=0, then:

$$\cancel{V}^{0} - \frac{q}{C} - iR = 0$$

$$-\frac{Q}{C} - iR = 0$$

$$i = -\frac{Q}{RC} = -\frac{\cancel{C}V}{R\cancel{C}}$$

$$\boxed{I_0 = -\frac{V}{R}}$$

current @ t=0

At some later time "t" the charge on capacitor is "q" and loop rule gives:

$$\boxed{-iR - \frac{q}{C} = 0}$$

$$\frac{dq}{dt} R = -\frac{q}{C}$$

$$\frac{dq}{dt} = -\frac{q}{RC}$$

$$\int_Q^q \frac{dq}{q} = -\frac{1}{RC}\int_0^t dt$$

$$\ln q \Big|_Q^q = -\frac{1}{RC} t$$

$$\ln\left(\frac{q}{Q}\right) = -\frac{1}{RC} t$$

$$\frac{q}{Q} = e^{-t/RC}$$

$$C = \frac{q}{V}$$

$$\boxed{q(t) = Q e^{-t/RC}}$$

$$i(t) = \frac{dq}{dt} = -\frac{Q}{RC} e^{-t/RC} = -\frac{\cancel{C}V}{R\cancel{C}} e^{-t/RC}$$

$$\boxed{i(t) = I_0 e^{-t/RC}}$$

$$V(t) = \frac{q}{C} = \frac{Q}{C} e^{-t/RC}$$

$$\boxed{V(t) = V e^{-t/RC}}$$

$$q(\tau) = 0.37Q$$

$$V(\tau) = 0.37V$$